MAT2705-04/05 Final Exam 23S

a) $$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}'' = \underbrace{\begin{bmatrix} -10 & 3 \\ 2 & -15 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 30\cos 2t \\ 30\cos 2t \end{bmatrix}}_{F}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}, \quad \begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} -13 \\ 5 \end{bmatrix}$$

b) $$0 = |A - \lambda I| = \begin{vmatrix} -10-\lambda & 3 \\ 2 & -15-\lambda \end{vmatrix} = (\lambda+10)(\lambda+15) - 6 = \lambda^2 + 25\lambda + 150$$

$$\underset{\text{Maple}}{=} (\lambda+9)(\lambda+16) = 0 \rightarrow \lambda = -9, -16$$

$\lambda_1 = -9$:

$$A + 9I = \begin{bmatrix} -10+9 & 3 \\ 2 & -15+9 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 2 & -6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 - 3x_2 = 0$

$0 = 0$

x_1 x_2

L F $\rightarrow x_2 = t \rightarrow x_1 = 3t$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix} = t \underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{= b_1}$$

$\lambda_2 = -16$:

$$A + 16I = \begin{bmatrix} -10+16 & 3 \\ 2 & -15+16 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 + \frac{1}{2} x_2 = 0$

$0 = 0$

L F

x_1 x_2 $\rightarrow x_2 = t \rightarrow x_1 = -\frac{1}{2} t$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1/2\, t \\ t \end{bmatrix} = \frac{t}{2} \underbrace{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}_{b_2}$$

$|\lambda_1| < |\lambda_2|$

$$\boxed{B = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} \quad \lambda = -9, -16}$$

$$B^{-1} = \frac{1}{6+1} \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} = \boxed{\frac{1}{7} \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}}$$

$$F_B = B^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos 2t = \frac{\cos 2t}{7} \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \boxed{\frac{\cos 2t}{7} \begin{bmatrix} 3 \\ 2 \end{bmatrix}}$$

$$A_B = B^{-1} A B = \begin{bmatrix} -9 & 0 \\ 0 & -16 \end{bmatrix}$$

c) $$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = y_1 b_1 + y_2 b_2 = B \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$x'' = Ax + F$$

$$B^{-1}[\,(By)'' = A(By) + F\,]$$

$$y'' = \underbrace{B^{-1}AB}_{A_B}\, y + \underbrace{B^{-1}F}_{F_B}$$

c) $$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}'' = \begin{bmatrix} -9 & 0 \\ 0 & -16 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \frac{30\cos 2t}{7} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$y_1'' = -9y_1 + \frac{30 \cdot 3}{7}\cos 2t \qquad y_1'' + 9y_1 = 30\left(\tfrac{3}{7}\right)\cos 2t$$

$$y_2'' = -16y_2 + \frac{30 \cdot 2}{7}\cos 2t \qquad y_2'' + 16y_2 = 30\left(\tfrac{2}{7}\right)\cos 2t$$

d) $$y_{1h} = c_1\cos 3t + c_2 \sin 3t$$

$$y_{2h} = c_3 \cos 4t + c_4 \sin 4t$$

e) $$y_{1p} = c_5 \cos 2t \qquad y_{1p}'' + 9y_{1p} = \overbrace{(-4+9)}^{5}\, c_5 \cos 2t = 30\left(\tfrac{3}{7}\right)\cos 2t$$

$$y_{2p} = c_6 \cos 2t \qquad y_{2p}'' + 16y_{2p} = \underbrace{(-4+16)}_{=12}\, c_6 \cos 2t = 30\left(\tfrac{2}{7}\right)\cos 2t$$

$$5c_5 = \left(\tfrac{3}{7}\right)30,\ c_5 = \frac{18}{7} \qquad y_{1p} = \frac{18}{7}\cos 2t \qquad y_1 = y_{1h} + y_{1p}$$

$$12c_6 = \left(\tfrac{2}{7}\right)30,\ c_6 = \frac{5}{7} \qquad y_{2p} = \frac{5}{7}\cos 2t \qquad y_2 = y_{2h} + y_{2p}$$

f) $$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left(c_1\cos 3t + c_2 \sin 3t + \tfrac{18}{7}\cos 2t\right) b_1 + \left(c_3\cos 4t + c_4 \sin 4t + \tfrac{5}{7}\cos 2t\right) b_2$$

$$= \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_1\cos 3t + c_2\sin 3t + 18/7\cos 2t \\ c_3\cos 4t + c_4 \sin 4t + 5/7 \cos 2t \end{bmatrix}$$

g) $$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -3c_1 \sin 3t + 3c_2 \cos 3t - 36/7 \sin 2t \\ -4c_3 \sin 3t + 4c_4 \cos 4t - 10/7 \sin 2t \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 + 18/7 \\ c_3 + 5/7 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \end{bmatrix} \qquad \begin{bmatrix} c_1 + 18/7 \\ c_3 + 5/7 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 7 \\ 11 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 25 \\ 26 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 25-18 \\ 26-5 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 7 \\ 21 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3c_2 \\ 4c_4 \end{bmatrix} = \begin{bmatrix} -13 \\ 5 \end{bmatrix} \qquad \begin{bmatrix} 3c_2 \\ 4c_4 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -13 \\ 5 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -21 \\ 28 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} c_2 \\ c_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (\cos 3t - \sin 3t)\, b_1 + (3\cos 4t + \sin 4t)\, b_2 + \frac{\cos 2t}{7} \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 18 \\ 5 \end{bmatrix}$$

$$= \frac{1}{7}\begin{bmatrix} 49 \\ 28 \end{bmatrix} = \boxed{\begin{bmatrix} 7 \\ 4 \end{bmatrix} = b_3}$$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (\cos 3t - \sin 3t)\begin{bmatrix} 3 \\ 1 \end{bmatrix} + (3\cos 4t + \sin 4t)\begin{bmatrix} -1 \\ 2 \end{bmatrix} + \cos 2t \begin{bmatrix} 7 \\ 4 \end{bmatrix}}$$

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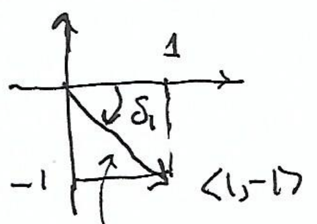
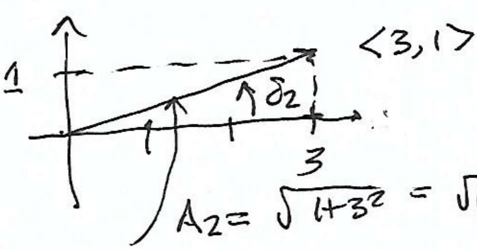
g) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3(\cos 3t - \sin 3t) - (3\cos 4t + \sin 4t) + 7\cos 2t \\ \cos 3t - \sin 3t + 2(3\cos 4t + \sin 4t) + 4\cos 2t \end{bmatrix}$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3\cos 3t - 3\sin 3t - 3\cos 4t - \sin 4t + 7\cos 2t \\ \cos 3t - \sin 3t + 6\cos 4t + 2\sin 4t + 4\cos 2t \end{bmatrix}}$$

so $x_1 = \cdots$
$x_2 = \cdots$

h) $y_{1h} = \cos 3t - \sin 3t$ $\qquad$ $y_{2h} = 3\cos 4t + \sin 4t$





$A_1 = \sqrt{1+1} = \sqrt{2} \approx 1.41$ $\qquad$ $A_2 = \sqrt{1+3^2} = \sqrt{10}$

$\delta_1 = -\arctan(1)$ $\qquad$ $\delta_2 = \arctan 3$

$= -\pi/4$

$y_{1h} = \sqrt{2}\cos(3t + \pi/4)$ $\qquad$ $y_{2h} = \sqrt{10}\cos(4t - \arctan 3)$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \cos(3t + \pi/4)\cdot\sqrt{2}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \cos(4t - \arctan 3)\sqrt{10}\begin{bmatrix} -1 \\ 2 \end{bmatrix} + \cos 2t\begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

i) parallelogram sides: $\pm y_1 = \sqrt{2} \approx 1.414$ $\qquad$ $\pm y_2 = \sqrt{10} \approx 3.162$

$\pm b_3 = \langle 7,4\rangle$, $b_1 = \langle 3,1\rangle$, $b_2 = \langle -1,2\rangle$

(see Maple)

plane of each sinsoidal function indicating your angles, and where b_3 is the vector coefficient of $\cos(2\,t)$ that corresponds to the particular solution in response to the driving force. You may chose either range for the phase shift evaluation but make sure your angle is labeled correctly in your diagrams.

i) On the grid provided plot the two eigenvectors and the new coordinate axes (label everything), and then the 6 vectors $\pm A_1\,b_1, \pm A_2\,b_2, \pm b_3$

and finally the parallelogram whose sides correspond to the lines $y_1 = \pm A_1, y_2 = \pm A_2$. [This parallelogram confines the homogeneous solution.]