MAT2705-04/05 23S Final Exam   Print Name (Last, First) _____|__
Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical
notation, identifying expressions by their proper symbols (introducing them if necessary), and use equal signs and
arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem.
Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and
what type (Maple, GC). **You may use technology for row reductions, determinants and matrix inverses
without showing details (identify technology).** Otherwise only use technology to CHECK hand calculations, not
subsitute for them, unless specifically requested. Remember "exact" means no decimals!

Consider the initial value problem:
$$x_1'' = -10\,x_1 + 3\,x_2 + 30\cos(2\,t),$$
$$x_2'' = \quad 2\,x_1 - 15\,x_2 + 30\cos(2\,t)\,,$$
$$x_1(0) = 7, x_2(0) = 11,$$
$$x_1'(0) = -13, x_2'(0) = 5$$

Solve this problem following these steps. To make sure you copy down the numbers correctly in working this
problem, know that the final expressions for the unknowns only involve single digit integers, although you
encounter fractions along the way.

$$x_1(0) = 7, x_2(0) = 11, \mathrm{D}(x_1)(0) = -13, \mathrm{D}(x_2)(0) = 5 \tag{1}$$

a) Write this system of DEs AND initial conditions in matrix form, identifying the coefficient matrix A. and the
driving force F.

b) Find the eigenvalues and eigenvectors of the coefficient matrix by hand showing all steps in the matrix solution
process (identifying leading and free variables, stating the reduced matrix equations, etc.) and order the
eigenvalues by increasing absolute value $|\lambda_1| < |\lambda_2|$ and **choose the smallest integer component eigenvectors** in
the basis changing matrix $B = \langle b_1 | b_2 \rangle$. State its inverse B^{-1}. and the diagonalized matrix $A_B = B^{-1} A B$ and
calculate by hand $F_B = B^{-1} F$.

c) Introduce the decoupling change of variables $x = B\,y = y_1\,b_1 + y_2\,b_2$ and re-express the matrix DE in the new
variables, and then write down the corresponding decoupled scalar DEs for the new variables y_1, y_2.

d) Write down the general homogenous solutions of each decoupled DE.

e) Find the particular solution of each decoupled DE using the method of undetermined coefficients.

f) Write down the general solution for the original variables in matrix or vector form (as a linear combination) or in
scalar form as you wish.

g) Impose the initial conditions on this general solution and express the resulting solution in scalar form for each of
the unknowns.

h) Now rewrite your final solution in the following linear combination vector form by re-expressing the
homogeneous solutions y_{1h}, y_{2h} in exact phase-shifted cosine form

$$x = A_1 \cos\left(\omega_1 t - \delta_1\right) b_1 + A_2 \cos\left(\omega_2 t - \delta_2\right) b_2 + \cos(2\,t)\, b_3\,,$$

supporting your two amplitude-phase-shift calculations with clearly labeled triangle diagrams in the coefficient plane of each sinsoidal function indicating your angles, and where b_3 is the vector coefficient of $\cos(2\,t)$ that corresponds to the particular solution in response to the driving force. You may chose either range for the phase shift evaluation but make sure your angle is labeled correctly in your diagrams.

i) On the grid provided plot the two eigenvectors and the new coordinate axes (label everything), and then the 6 vectors $\pm A_1\, b_1, \pm A_2\, b_2, \pm b_3$

and finally the parallelogram whose sides correspond to the lines $y_1 = \pm A_1$, $y_2 = \pm A_2$. [This parallelogram confines the homogeneous solution.]