MAT2705-04/05 23S Test 3

① a) $x'' + 4x' + 5x = 10\cos 3t$

$x = e^{rt} \rightsquigarrow (r^2 + 4r + 5)e^{rt} = 0$

$r^2 + 4r + 5 = 0$

$r = \dfrac{-4 \pm \sqrt{16 - 4 \cdot 5}}{2} = -2 \pm i$

$e^{rt} = e^{(-2 \pm i)t} = e^{-2t} e^{\pm i t}$

$= e^{-2t}(\cos t \pm i \sin t)$

$\therefore \boxed{x_h = e^{-2t}(c_1 \cos t + c_2 \sin t)}$ homogeneous soln

$5\,[x_p = c_3 \cos 3t + c_4 \sin 3t$

$+4\,[x_p' = -3c_3 \sin 3t + 3c_4 \cos 3t$

$+1\,[x_p'' = -9c_3 \cos 3t - 9c_4 \sin 3t$

$x_p'' + 4x_p' + 5x_p = [(5-9)c_3 + 12c_4]\cos 3t$
$\quad + [-12c_3 + (5-9)c_4]\sin 3t$

$= \underbrace{(-4c_3 + 12c_4)}_{=10}\cos 3t + \underbrace{(-12c_3 - 4c_4)}_{=0}\sin 3t = 10\cos 3t$

$$\begin{bmatrix} -4 & 12 \\ -12 & -4 \end{bmatrix}\begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \frac{1}{\underbrace{16+144}_{=160}}\begin{bmatrix} -4 & -12 \\ 12 & -4 \end{bmatrix}\begin{bmatrix} 10 \\ 0 \end{bmatrix} = \frac{10}{160}\begin{bmatrix} -4 \\ 12 \end{bmatrix}$$

$= \frac{1}{4}\begin{bmatrix} -1 \\ 3 \end{bmatrix} \to \boxed{x_p = \frac{1}{4}(-\cos 3t + 3\sin 3t)}$ particular soln

$\boxed{x = x_h + x_p = e^{-2t}(c_1 \cos t + c_2 \sin t) + \frac{1}{4}(-\cos 3t + 3\sin 3t)}$ gen soln

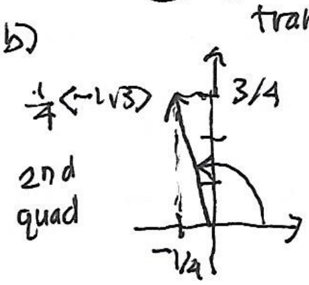
$x' = -2e^{-2t}(c_1 \cos t + c_2 \sin t) + \frac{1}{4}(3\sin 3t + 9\cos 3t)$
$\quad + e^{-2t}(-c_1 \sin t + c_2 \cos t)$

$x(0) = c_1 - \frac{1}{4} = 0 \to c_1 = 1/4$

$x'(0) = -2c_1 + c_2 + \frac{9}{4} = 0 \to c_2 = -9/4 + 2(\frac{1}{4}) = -\frac{7}{4}$

$\boxed{x = \underbrace{\frac{1}{4}e^{-2t}(\cos t - 7\sin t)}_{\text{transient } x_t} + \underbrace{\frac{1}{4}(-\cos 3t + 3\sin 3t)}_{\text{steady state } x_{ss}}}$ IVP soln

b)



$A = \frac{1}{4}\sqrt{1+9} = \frac{\sqrt{10}}{4}$

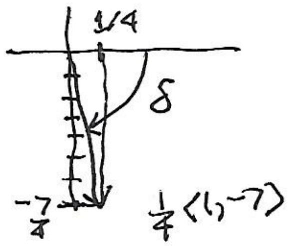
$\delta = \pi - \arctan 3$

$\boxed{x_{ss} = \underbrace{\frac{\sqrt{10}}{4}}_{\approx 0.790}\cos(3t - \pi + \arctan 3)}$

MAT2705-04/05 23S Test 3 (2)

① c) $x_t = \frac{1}{4} e^{-2t}(\cos t - 7\sin t) = \frac{\sqrt{50}}{4} e^{-2t} \cos(t + \arctan 7)$

$\tau = \frac{1}{2}$



$A = \frac{1}{4}\sqrt{1+49} = \frac{\sqrt{50}}{4}$

$\delta = -\arctan 7$

d) $5\tau = \frac{5}{2} = 2.5$

d) $x = \pm \frac{\sqrt{50}}{4} e^{-2t}$ envelope curves

$t + \arctan 7 = \pi \Leftrightarrow \cos(t + \arctan 7) = -1$

$t = \pi - \arctan 7 \approx 1.713$ (first value for $t > 0$)

d) e) plots see Maple worksheet.

② $9x'' + 6x' + 37x = \cos(\omega t)$

$$x'' + \underbrace{\tfrac{2}{3}}_{k_0} x' + \underbrace{\tfrac{37}{9}}_{\omega_0^2} x = \underbrace{\tfrac{1}{9}}_{B_0} \cos \omega t$$

$k_0 = \frac{2}{3}$

$\tau_0 = 1/k_0 = \frac{3}{2} = 1.5$

$\omega_0 = \sqrt{\frac{37}{9}} = \frac{\sqrt{37}}{3} \approx 2.0276$

$Q = \omega_0 \tau_0 = \frac{3}{2}\frac{\sqrt{37}}{3} = \frac{\sqrt{37}}{2} \approx \boxed{3.0419}$

$B_0 = 1/9$

ω_{peak} is slightly smaller: $\omega_{peak} \lesssim \omega_0$

$$A(\omega) = \frac{1}{9}\left(\left(\omega^2 - \tfrac{37}{9}\right)^2 + \tfrac{4}{9}\omega^2\right)^{-1/2}$$

$$0 = A'(\omega) = \frac{1}{9}(\cdots)^{-3/2}\left(2\left(\omega^2 - \tfrac{37}{9}\right)(2\omega) + \tfrac{8}{9}\omega\right) = 4\omega\Big(\underbrace{\omega^2 - \tfrac{37}{9} + \tfrac{2}{9}}_{\omega^2 - \frac{35}{9}}\Big) = 0$$

$\omega = 0, \boxed{\frac{\sqrt{35}}{3} \approx 1.9720 = \omega_{peak}}$

$$A(0) = \frac{1/9}{\sqrt{\left(\frac{37}{9}\right)^2}} = \frac{1}{37} \approx 0.0270$$

$$A(\omega_0) = \frac{1/9}{\sqrt{\left(\frac{37}{9} - \frac{37}{9}\right)^2 + \frac{4}{9}\left(\frac{37}{9}\right)}} = \frac{1}{\sqrt{4(37)}} = \frac{1}{2\sqrt{37}} \approx 0.0822$$

$$A(\omega_{peak}) = \frac{1/9}{\sqrt{\left(\frac{35}{9} - \frac{37}{9}\right)^2 + \frac{4}{9}\left(\frac{35}{9}\right)}} = \frac{1}{\sqrt{4 + 4(35)}} = \frac{1}{2 \cdot 6} = \frac{1}{12} \approx 0.0833$$

$$\frac{A(\omega)}{A(0)} = \frac{37/9}{\sqrt{(\omega^2 - 37/9)^2 + \frac{4}{9}\omega^2}}$$

$$\frac{A(\omega_{peak})}{A(0)} = \frac{1/12}{1/37} = \frac{37}{12} \approx \boxed{3.0833} \gtrsim 3.0419$$

slightly larger

3 a) $x_1' = -4x_1 - x_2 - x_3$
$x_2' = -x_1 - 4x_2 - x_3$
$x_3' = -x_1 - x_2 - 4x_3$
$x_1(0) = 3,\ x_2(0) = -6,\ x_3(0) = 9$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = \underbrace{\begin{bmatrix} -4 & -1 & -1 \\ -1 & -4 & -1 \\ -1 & -1 & -4 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ 9 \end{bmatrix}$$

b) $$0 = |A - \lambda I| = \begin{vmatrix} -4-\lambda & -1 & -1 \\ -1 & -4-\lambda & -1 \\ -1 & -1 & -4-\lambda \end{vmatrix} \overset{\text{Maple}}{=} -(\lambda^3 + 12\lambda^2 + 45\lambda + 54)$$

$\hookrightarrow \lambda = \underbrace{-3, -3}_{m=2}, -6$

c)
$\lambda = -3$:

$$A + 3I = \begin{bmatrix} -4+3 & -1 & -1 \\ -1 & -4+3 & -1 \\ -1 & -1 & -4+3 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \longrightarrow \begin{array}{c} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} x_1 & x_2 & x_3 \\ L & F & F \end{matrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_2 = t_1$
$x_3 = t_2$

$x_1 + x_2 + x_3 = 0 \to x_1 = -t_1 - t_2$
$0 = 0$
$0 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t_1 - t_2 \\ t_1 \\ t_2 \end{bmatrix} = t_1 \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{=b_1} + t_2 \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{=b_2}$$

$\lambda = -6$:

$$A + 6I = \begin{bmatrix} -4+6 & -1 & -1 \\ -1 & -4+6 & -1 \\ -1 & -1 & -4+6 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ 2 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & -3 & 3 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{array}{c} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} x_1 & x_2 & x_3 \\ L & L & F \end{matrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\longrightarrow x_3 = t$

$x_1 - x_3 = 0$
$x_2 - x_3 = 0$
$0 = 0$

$x_1 = t$
$x_2 = t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{=b_3}$$

$$\begin{array}{rccc} \lambda = & -3 & -3 & -6 \end{array}$$
$$B = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

d) $$A_B = B^{-1} A B = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

③ e) $x = By \to x' = Ax \to (By)' = ABy \to By' = ABy \to$

$y' = B^{-1}ABy = A_B y$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}' = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -6 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -3y_1 \\ -3y_2 \\ -6y_3 \end{bmatrix} \qquad \begin{matrix} y_1' = -3y_1 \\ y_2' = -3y_2 \\ y_3' = -6y_3 \end{matrix} \qquad \begin{matrix} y_1 = c_1 e^{-3t} \\ y_2 = c_2 e^{-3t} \\ y_3 = c_3 e^{-6t} \end{matrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 e^{-3t} \\ c_2 e^{-3t} \\ c_3 e^{-6t} \end{bmatrix} = \begin{bmatrix} -c_1 e^{-3t} - c_2 e^{-3t} + c_3 e^{-6t} \\ c_1 e^{-3t} + c_3 e^{-6t} \\ c_2 e^{-3t} + c_3 e^{-6t} \end{bmatrix}$$ general soln

f) $$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = B\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ 9 \end{bmatrix} \to \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = B^{-1}\begin{bmatrix} 3 \\ -6 \\ 9 \end{bmatrix}$$

$$= \underset{\uparrow\ \text{maple}}{\frac{1}{3}\begin{bmatrix} -1 & 2 & -1 \\ -1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}}\begin{bmatrix} 3 \\ -6 \\ 9 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -3-12-9 \\ -3+6+18 \\ 3-6+9 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -24 \\ 21 \\ 6 \end{bmatrix} = \begin{bmatrix} -8 \\ 7 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8e^{-3t} - 7e^{-3t} + 2e^{-6t} \\ -8e^{-3t} + 2e^{-6t} \\ 7e^{-3t} + 2e^{-6t} \end{bmatrix} = \begin{bmatrix} e^{-3t} + 2e^{-6t} \\ -8e^{-3t} + 2e^{-6t} \\ 7e^{-3t} + 2e^{-6t} \end{bmatrix}$$

g) $\underbrace{\tau_1 = \frac{1}{3}}_{\text{takes longer to decay}} > \tau_2 = \frac{1}{6}$

$5\tau_1 = \frac{5}{3} = 1\frac{2}{3} \approx 1.67$