MAT1505-02/03 23F Quiz 7 Print Name (Last, First) ____________________ , ________________|____

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). INDICATE where technology is used and what type (Maple, GC). **Technology can only be used to check hand calculations and not substitute for them, unless specifically stated.** Numerical values can be evaluated with technology.

1. a) Use the comparison ~~theorem~~ test to show that $\sum_{n=1}^{\infty} \frac{1}{5+n^5}$ converges. Explain in detail.

b) Use the first 10 terms of $\sum_{n=1}^{\infty} \frac{1}{5+n^5}$ to approximate the sum of the series, giving the result to 4 decimal places.

c) Using the obvious comparison series $\sum_{n=1}^{\infty} b_n$, estimate the error (to 2 significant figures) in your approximation in part b) using the integral test approximation for the remainder of this new series when truncating it after 10 terms.

d) Using that same approach to estimate the truncation error, for what value of n will the partial sum S_n be guaranteed to approximate the series sum to an error lower than 10^{-8}?

2. a) Show that $\sum_{n=1}^{\infty} \frac{2n^2}{3n^6+1}$ is convergent using the limit comparison test.

b) Show that the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n^2}{3n^6+1}$ is convergent using the alternating series test.
[Hint: use a derivative to show that $|a_n|$ is a decreasing sequence for $n \geq 1$. Where is the derivative zero? Where is it negative? Does a plot confirm this?]

3. a) Use Maple to evaluate $\sum_{n=1}^{\infty} \left(\frac{(-1)^{n-1}}{n^2 2^n} \right)$ giving your numerical value to 4 decimal places.

b) How many terms are needed to approximate this series to an error less than 0.00005 in absolute value? Justify your answer using the alternating series truncation estimate.

solution

① a) $\underbrace{\sum_{n=1}^{\infty} \frac{1}{5+n^5}} < \sum_{n=1}^{\infty} \frac{1}{0+n^5} = \sum_{n=1}^{\infty} \frac{1}{n^5}$ this is a convergent p-series with $p=5>1$.

So this converges by the comparison test: $b_n = \frac{1}{n^5}$.

b) $S_{10} = \sum_{n=1}^{10} \frac{1}{5+n^5} \underset{\text{Maple}}{=} 0.1992627 \approx \boxed{0.1993}$

c) $|\text{Truncation error}| < \int_{10}^{\infty} x^{-5}\,dx = \lim_{N\to\infty} \int_{10}^{N} x^{-5}\,dx = \lim_{N\to\infty} \frac{x^{-4}}{-4}\Big|_{10}^{N}$

round up to satisfy inequality

$= \lim_{N\to\infty} \left(\underbrace{-\cancel{\frac{1}{4N^4}}}_{\to 0} + \frac{1}{4\cdot 10^4} \right) = .25\cdot 10^{-4} = \boxed{.000025}$

d) $|\text{truncation error}| < \int_{n}^{\infty} x^{-5}\,dx = \cdots = \lim_{N\to\infty} \left(\underbrace{-\cancel{\frac{1}{4N^4}}}_{0} + \frac{1}{4n^4} \right) = \frac{1}{4n^4} < 10^{-8}$

$\frac{1}{4} < 10^{-8} n^4 \to n^4 > \frac{1}{4}\cdot 10^{8} \to n > \frac{1}{4^{1/4}} \underbrace{10^{(8)1/4}}_{10^2} \approx 70.7 \to \boxed{71 = n}$

MAT1505-02/03 23F Quiz 7

(2) a) $$\sum_{n=1}^{\infty} \frac{2n^2}{3n^6+1}$$

looks like $\frac{2n^2}{3n^6} = \frac{2}{3n^4}$ for large n, so use this in the limit comparison test.

careful response:

$$\lim_{n\to\infty} \frac{\frac{2n^2}{3n^6+1}}{\left(\frac{2}{3n^4}\right)} = \lim_{n\to\infty} \frac{2n^2}{3n^6+1}\left(\frac{3n^4}{2}\right) = \lim_{n\to\infty} \frac{3n^6 \;/n^6}{(3n^6+1)/n^6}$$

$$= \lim_{n\to\infty} \frac{3}{3+\underbrace{\frac{1}{n^6}}_{\to 0}} = \frac{3}{3} = 1$$ same limiting behavior as $n\to\infty$

intuitive response:

$\sum_{n=1}^{\infty} \frac{2}{3n^4}$ has same limiting behavior as $n\to\infty$ clearly! (only highest powers count)

and this is proportional to a $p=4$ series with $p>1$ so it converges, implying the convergence of the original series.

b) $$\sum_{n=1}^{\infty} (-1)^{n-1} \underbrace{\frac{2n^2}{3n^6+1}}_{|a_n| = b_n}$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{2n^2/n^2}{(3n^6+1)/n^2} = \lim_{n\to\infty} \frac{2}{3n^4+\frac{1}{n^2}} = 0$$

$$f(x) = \frac{2x^2}{3x^6+1} \qquad f'(x) \underset{\text{Maple}}{=} \frac{4x-24x^7}{(3x^6+1)^2} = 0 \to 4x(1-6x^6) = 0$$

$x^6 = \frac{1}{6}$

$x = \left(\frac{1}{6}\right)^{1/6} = \frac{1}{6^{1/6}} < 1$

≈ 0.74

clearly negative when $x^6 > \frac{1}{6}$
confirmed by plot of $f'(x)$
so $f(x)$ is a decreasing function for $x>1$
so b_n is a decreasing sequence.

Then this series converges by the alternating series test.

(3) a) $$\sum_{n=1}^{\infty} (-1)^{n-1} \underbrace{\frac{1}{n^2 2^n}}_{= b_n} \underset{\text{Maple}}{=} (\dots) \approx 0.4484142 \approx \boxed{0.4484}$$

b) error in using the partial sum $\sum_{k=1}^{n} (-1)^{k-1} \frac{1}{k^2 2^k}$ is less in absolute value than the next term in the series

$$|\text{Error}| < \frac{1}{(n+1)^2\, 2^{n+1}} < 0.00005 \to (n+1)^2 2^{n+1} > \frac{1}{0.00005} = 2\times 10^4$$

solve $(n+1)^2 2^{n+1} = 2\times 10^4$ numerically $\to n = 7.21 \xrightarrow{\text{round up}} \boxed{n=8}$