

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). INDICATE where technology is used and what type (Maple, GC). **Technology can only be used to check hand calculations and not substitute for them, unless specifically stated.**

1. The probability distribution $p(x) = c \cdot (a - x) x^2 = c \cdot (a x^2 - x^3)$ is an example of a beta distribution for $0 \leq x \leq a$.

a) For what value of c is the total probability equal to 1:

$$P(0 \leq x \leq a) = \int_0^a p(x) dx = 1? \text{ Set } c \text{ equal to this value for the remainder of the problem.}$$

b) Evaluate (by hand) the expected value $\mu_x = \int_0^a x p(x) dx$ of the variable x .

c) By changing the independent variable from x to $u = \frac{x}{a}$ (dimensionless!) on the new interval $0 \leq u \leq 1$, show that the middle equality holds (the outer parts are the definitions of probability):

$$P(x_1 \leq x \leq x_2) = \int_{x_1}^{x_2} p(x) dx = \int_{u_1}^{u_2} 12 \cdot (1 - u) \cdot u^2 du = P(u_1 \leq u \leq u_2),$$

where $u_i = \frac{x_i}{a}$. This makes the new variable the "standard" beta variable for this problem, which applies to the

distribution of the fractional variable $u = \frac{x}{a}$. The new variable enables us to visualize the properties of this entire one-parameter family of distributions with a single standard graph without any parameter.

d) Find the exact value u_p at which this standard distribution $p_s(u) = 12 \cdot (1 - u) \cdot u^2$ has its peak value (calc 1 technique!) and state the value of u_p to 3 decimal places.

e) Using your part b) result, evaluate the exact expected value $\mu = \frac{\mu_x}{a}$ and its value to 3 decimal places.

f) Find the median value u_m of the standard variable by solving

$$\int_0^{u_m} 12 \cdot (1 - u) \cdot u^2 du = \frac{1}{2} \text{ numerically on the unit interval to 3 decimal places.}$$

g) Order the three values u_m, u_p, μ from smallest to largest and indicate them on the axis of a technology plot of the standard distribution function $y = p_s(u)$ on the unit interval, with a vertical line up to the corresponding point on the distribution function graph to better see where they fall on that distribution (use a ruler, annotate it labeling it sufficiently, print it out and attach).

2. Physicist Stephen Hawking, when asked his IQ, he replied: "I have no idea. People who boast about their IQ are losers." The IQ is a controversial number x but its distribution on the interval $0 \leq x < \infty$ is described by a normal

distribution with mean value 100 and standard deviation 15: $p(x) = \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}$. If one searches for what level

constitutes a "genius" (not a precisely defined concept), one finds 130 or 140 (sometimes 145). What percentages of the population have IQs greater than 130 and 140 respectively (to the nearest tenth of a percent)? Respond in a complete English sentence. How many genius's would there be in the current US population of about 332 million at these two cutoffs (2 significant figures)?