MAT1505-02/03 23F Final Exam (1)

① a) $x = \frac{1}{2}(t^2+3)$ $\quad \frac{dx}{dt} = \frac{1}{2}(2t+0) = t$

$y = 5 - \frac{1}{3}t^3$ $\quad \frac{dy}{dt} = 0 - \frac{1}{3}(3t^2) = -t^2$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-t^2}{t} = -t$$

$\left.\frac{dy}{dx}\right|_{t=1} = -1 = m$ $\quad x(1) = \frac{1}{2}(1+3) = 2$

$y(1) = 5 - \frac{1}{3}(1) = \frac{15-1}{3} = \frac{14}{3}$

$y - y(1) = m(x - x(1))$

$y - \frac{14}{3} = -1(x-2) \longrightarrow y = \frac{14}{3} - (x-2) = \frac{14}{3} + 2 - x = \frac{20}{3} - x$

$$\boxed{y = \frac{20}{3} - x}$$

b) $x'^2 + y'^2 = t^2 + (-t^2)^2 = t^2 + t^4 = t^2(1+t^2)$

$\sqrt{x'^2+y'^2} = \sqrt{t^2(1+t^2)} = t\sqrt{1+t^2}$ since $t \geq 0$

$$\boxed{L = \int_0^3 t\sqrt{1+t^2}\,dt}$$

$u = 1+t^2$

$du = 2t\,dt$

$\frac{1}{2}du = t\,dt$

$\int t(1+t^2)^{1/2}\,dt = \int u^{1/2}\frac{du}{2} = \frac{1}{2}\frac{u^{3/2}}{3/2} + C$

$= \frac{1}{3}(1+t^2)^{3/2} + C$

$$L = \frac{1}{3}(1+t^2)^{3/2}\Big|_0^3 = \frac{1}{3}\left((1+3^2)^{3/2} - (1+0)\right) = \boxed{\frac{1}{3}(10^{3/2}-1)}$$

$$= \boxed{\frac{1}{3}(10\sqrt{10}-1)}$$

② a)

$x = t^3 - 3t$ $\quad \frac{dx}{dt} = 3t^2 - 3 = 3(t^2-1)$

$y = t^3 - 3t^2$ $\quad \frac{dy}{dt} = 3t^2 - 6t = 3t(t-2)$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t(t-2)}{3(t^2-1)} = \frac{t(t-2)}{t^2-1}$$

numerator $= 0 \xrightarrow{H:} t = 0 \rightarrow x = 0, y = 0$

$t = 2 \rightarrow x = 2^3 - 3\cdot 2 = 2$

$y = 2^3 - 3\cdot 2^2 = -4$

denominator $= 0 \xrightarrow{V:} t = \pm 1 \rightarrow x = t(t^2-3)$

$= \pm 1(1-3) = \mp 2$

$y = t^2(t-3) = 1(\pm 1 - 3)$

$= -4, -2$

Horizontal:

$t=0$: $(0,0)$

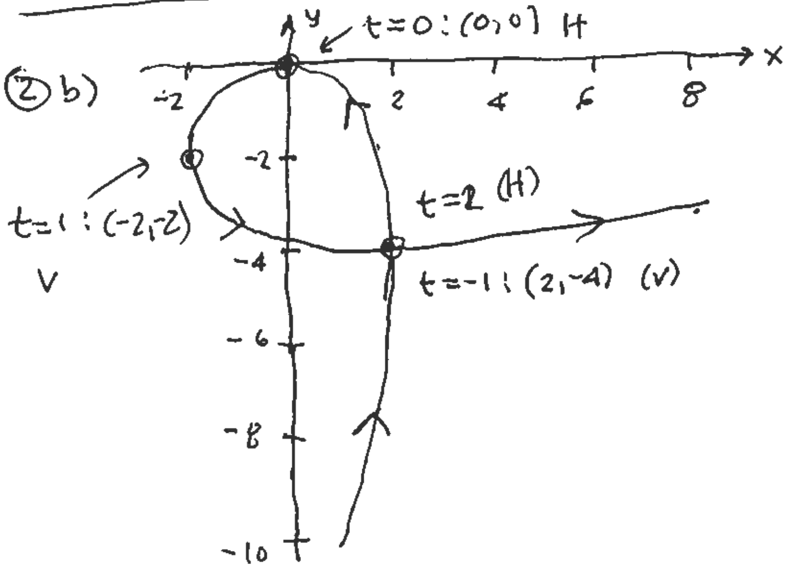
$t=2$: $(2,-4)$

vertical:

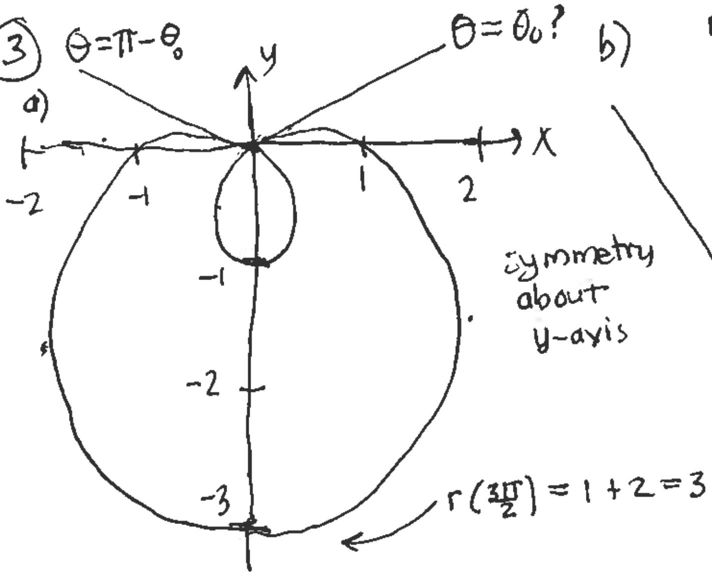
$t=-1$: $(2,-4)$

$t=1$: $(-2,-2)$

(2) b)



(3) a)



starts at $\theta = 0 \to r = 1$

and r decreases to 0 as $\sin\theta$ increases

at $\theta = \theta_0$ in first quadrant.

Then it goes negative for the inner loop.

b) $r = 1 - 2\sin\theta = 0$

$1 = 2\sin\theta \to \sin\theta = \frac{1}{2}$

$\hookrightarrow \theta = \frac{\pi}{6},\ \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

$r \le 0$: $\boxed{\frac{\pi}{6} \le \theta \le \frac{5\pi}{6}}$

c) $A = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2} r^2 \, d\theta$

$= \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}(1-2\sin\theta)^2 \, d\theta$

$\boxed{= \int_{\frac{\pi}{6}}^{5\pi/6} \frac{1}{2}(1 - 4\sin\theta + 4\sin^2\theta)\, d\theta}$

d) $\underset{\text{Maple}}{=} \boxed{\pi - \frac{3\sqrt{3}}{2}}$

e) $\approx 0.543516442 \approx \boxed{0.544}$

$r = 1 - 2\sin\theta$

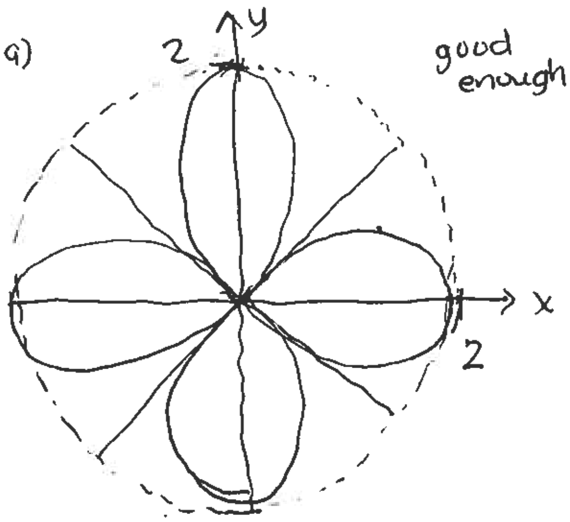
f) $r' = 0 - 2\cos\theta = -2\cos\theta$

$r^2 + r'^2 = (1-2\sin\theta)^2 + (-2\cos\theta)^2$

$= 1 - 4\sin\theta + \underbrace{4\sin^2\theta + 4\cos^2\theta}_{4} = 5 - 4\sin\theta$

$\boxed{L = \int_0^{2\pi} \sqrt{5 - 4\sin\theta}\, d\theta} \underset{g)}{=} 12 \text{ EllipticE}\left(\frac{2\sqrt{2}}{3}\right) \approx 13.36489322 \approx \boxed{13.365}$

4) a)



good enough!

b) $r = 2\cos(\underbrace{2\theta}_{})$

$= \frac{\pi}{2} \to \theta = \frac{\pi}{4}$

increasing θ from 0, r decreases to 0 by $\theta = \frac{\pi}{4}$ since $\cos(\frac{\pi}{2}) = 0$

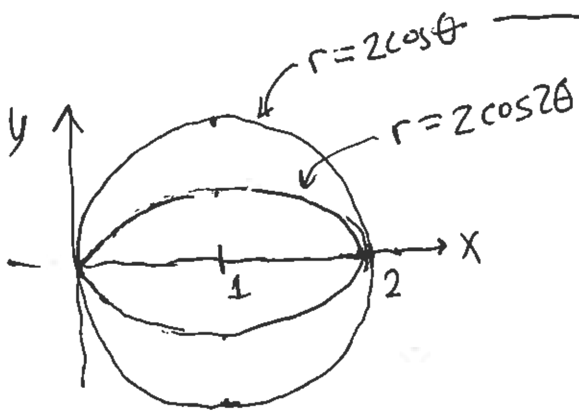
so by symmetry about the x-axis

$\boxed{-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}}$ corresponds to that right loop

c) $A = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} r^2 d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}(2\cos 2\theta)^2 d\theta = \boxed{\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2\cos^2 2\theta \, d\theta \text{ or } = 2\int_0^{\frac{\pi}{4}} \cos^2 2\theta \, d\theta}$

d) $\underset{\text{Maple}}{=} \boxed{\frac{\pi}{2} \approx 1.571}$

e)



Area $= \pi(1)^2 = \pi$

$\frac{A}{\text{Area}} = \frac{\pi/2}{\pi} = \boxed{\frac{1}{2} = 50\%}$

This looks about right.