MAT1505-02/0323F Takehome test 3

① $\underbrace{\sum_{n=0}^{\infty} \frac{3(x-6)^n}{(2n+1)\, 2^{2n+1}}}_{a_n}$ a) $\left|\frac{a_{n+1}}{a_n}\right| = \dfrac{\dfrac{3|x-6|^{n+1}}{(2(n+1)+1)\, 2^{2(n+1)+1}}}{\dfrac{3|x-6|^n}{(2n+1)\, 2^{2n+1}}} = \left(\frac{2n+1}{2n+3}\right) \frac{2^{2n+1}}{2^{2n+3}} \frac{|x-6|^{n+1}}{|x-6|^n}$

$= \frac{2n+1}{2n+3} \cdot \frac{1}{2^2} |x-6|$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \underbrace{\left(\frac{2n+1}{2n+3}\right)}_{\to 1} \frac{|x-6|}{4} = \frac{|x-6|}{4} < 1$ for convergence

$\hookrightarrow |x-6| < 4 = R$ radius of convergence.

endpoints: $|x-6| = 4$

$(x-6) = \pm 4 \to x = 6 \pm 4 = 2, 10$

$\underline{x=2,\ x-6=-4:}$

$\sum_{n=0}^{\infty} \frac{3(-4)^n}{(2n+1)\, 2^{2n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3 \cdot \overbrace{4^n}^{=1}}{(2n+1)\, 2^{2n} \cdot 2} = \sum_{n=0}^{\infty} (-1)^n \frac{3}{2(2n+1)}$

for large n, looks like: $\sum_{n=1}^{\infty} (-1)^n \frac{3}{4n}$ $\propto$ alternating harmonic series

converges by alternating series test

$\underline{x=10,\ x-6=4:}$

$\sum_{n=0}^{\infty} \frac{3 \cdot 4^n}{(2n+1)\, 2^{2n+1}} = \sum_{n=0}^{\infty} \frac{3 \cdot \overbrace{4^n}^{=1}}{(2n+1)\, 2^{2n} \cdot 2} = \sum_{n=0}^{\infty} \frac{3}{2(2n+1)}$

for large n, looks like $\sum_{n=1}^{\infty} \frac{3}{2n}$ $\propto$ harmonic series, diverges

so interval of convergence is: $\boxed{2 \le x < 10}$

② $\lim_{x\to 0} \frac{(1+2x^2)^{3/2} - 1 - 3x^2}{x^4} = \lim_{x\to 0} \frac{\cancel{1 + \frac{3}{2}(2x^2)} + \frac{\frac{3}{2}\left(\frac{1}{2}\right)}{1\cdot 2}(2x^2)^2 + \frac{\frac{3}{2}\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)}{1\cdot 2\cdot 3}(2x^2)^3 \cancel{- 1 - 3x^2}}{x^4}$ $\to 0$

$= \lim_{x\to 0} \frac{\frac{3}{2}x^4 - \frac{1}{2}x^6 + \dots}{x^4} = \lim_{x\to 0} \frac{3}{2} - \frac{1}{2}x^2 + \dots = \boxed{\frac{3}{2}}$

③ a) $f(x) = x^2 e^{-x^2/2} = x^2 \sum_{n=0}^{\infty} \frac{\left(-\frac{x^2}{2}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} \cdot x^2}{2^n n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{2^n n!}$

b) $\int f(x)\,dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{2^n n!}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{\int x^{2n+2}\,dx}{2^n n!}$

$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{2^n n!(2n+3)} + \cancel{C}$ 0

c) $= \frac{x^3}{2^0 0! \cdot 3} - \frac{x^5}{2^1 1! \cdot 5} + \frac{x^7}{2^2 2! \, 7} - \frac{x^9}{2^3 3! \cdot 9} + \cdots$

$= \frac{x^3}{3} - \frac{x^5}{10} + \frac{x^7}{56} - \frac{x^9}{432} + \cdots$

d) $\int_0^{0.4} x^2 e^{-x^2/2}\,dx = \left(\frac{(0.4)^3}{3} - \frac{(0.4)^5}{10} + \frac{(0.4)^7}{56} - \frac{(0.4)^9}{432} + \cdots \right) - 0$

$= \underbrace{0.0213333 - 0.0010024 + 0.0000293}_{\text{sum of first 3 terms:}} - \underbrace{6.1 \times 10^{-7}}_{< 10^{-6}} + \cdots$

f) 4th term less than target error

sum of first 3 terms: 0.0203386

$\approx \boxed{0.02034}$ to 5 decimal places, so yes this agrees with Maple's evaluation to 5 decimal places

④ a) $V = \frac{q}{r-x} - \frac{q}{r+x} = q\left[\frac{1}{r(1-\frac{x}{r})} - \frac{1}{r(1+\frac{x}{r})} \right]$

$= \frac{q}{r}\left[\sum_{n=0}^{\infty} \left(\frac{x}{r}\right)^n - \sum_{n=0}^{\infty} \left(\frac{-x}{r}\right)^n \right]$

$= \frac{q}{r}\left[\left(1 + \left(\frac{x}{r}\right) + \left(\frac{x}{r}\right)^2 + \left(\frac{x}{r}\right)^3 + \cdots\right) - \left(1 + \left(\frac{-x}{r}\right) + \left(\frac{-x}{r}\right)^2 + \left(\frac{-x}{r}\right)^3 + \cdots\right) \right]$

$= \frac{q}{r}\left[0 + \frac{2x}{r} + 0 + \frac{2x^3}{r^3} + \cdots \right] = \boxed{\frac{2qx}{r^2} + \frac{2qx^3}{r^4} + \cdots}$

b) $\frac{\frac{2q|x|^3}{|r|^4}}{\frac{2q|x|}{r^2}} = \frac{|x|^2}{|r|^2} < 0.01 = 10^{-2} \to \boxed{\left|\frac{x}{r}\right| < 10^{-1}} \to \boxed{|x| < 0.1\,|r|}$