MAT1505-02/03 23F Test 1B Take Home

① a) $\int f(x)\,dx = \int x^3\sqrt{4-x^2}\,dx = \int x^3 (u)^{1/2}\left(-\frac{du}{2x}\right) = -\frac{1}{2}\int x^2 u^{1/2}\,du$

$u = 4-x^2 \rightarrow x^2 = 4-u$

$du = -2x\,dx$

$-\frac{du}{2x} = dx$

$= -\frac{1}{2}\int (4-u)\,u^{1/2}\,du$ (conversion complete)

$= -\frac{1}{2}\int 4u^{1/2} - u^{3/2}\,du$ (expand, combine exponents)

$= -\frac{1}{2}\left(4\,\frac{u^{3/2}}{3/2} - \frac{u^{5/2}}{5/2}\right) + C$ (power rule)

$= -\frac{4}{3}u^{3/2} + \frac{1}{5}u^{5/2} + C$ (simplify)

$= \boxed{\frac{1}{5}(4-x^2)^{5/2} - \frac{4}{3}(4-x^2)^{3/2} + C}$

b) $f_{avg} = \frac{1}{2}\int_0^2 f(x)\,dx = \frac{1}{2}\left(\frac{1}{5}(4-x^2)^{5/2} - \frac{4}{3}(4-x^2)^{3/2}\right)\Big|_0^2$

$= \frac{1}{2}\left[0 - 0 - \left(\frac{1}{5}\cdot 4^{5/2} - \frac{4}{3}\cdot 4^{3/2}\right)\right]$

$= \frac{1}{2}\left[-\frac{1}{5}\underbrace{2^5}_{32} + \frac{4}{3}\underbrace{2^3}_{8}\right] = \frac{32}{2}\left[-\frac{1}{5} + \frac{1}{3}\right] = 16\left(\frac{5-3}{15}\right) = \boxed{\frac{32}{15}}$

≈ 2.13333

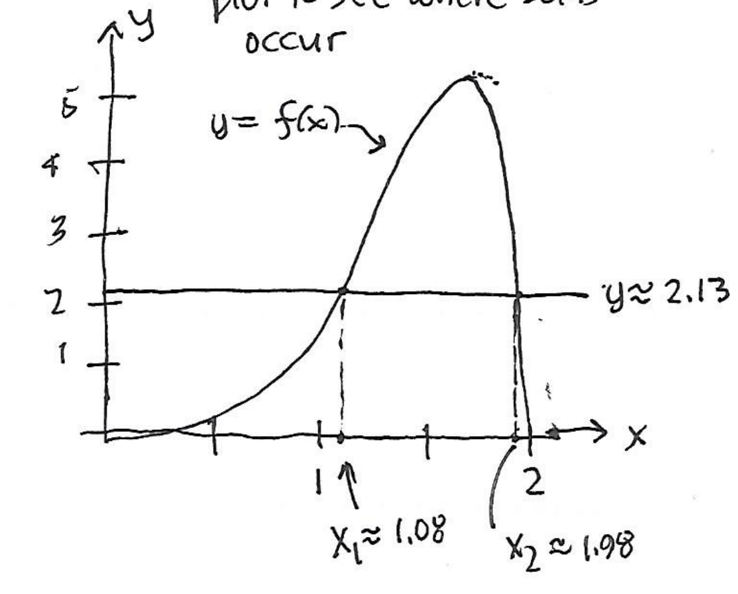
alternative simplification: factor!

$\int f(x)\,dx = (4-x^2)^{3/2}\left(\frac{1}{5}(4-x^2) - \frac{4}{3}\right) + C = (4-x^2)^{3/2}\,\frac{1}{15}\big(\underbrace{12-3x^2-20}_{-(8+3x^2)}\big) + C$

$= -\frac{(8+3x^2)(4-x^2)^{3/2}}{15} + C$

c) $x^3\sqrt{4-x^2} = \frac{32}{15} \approx 2.13333$

plot to see where solns occur



There will be 2 crossing points between 1 and 2, close to both endpoints.
In Maple do solve numerically from point $x=1$ and $x=2$ to get these.

$x_1 = 1.082519690 \approx \boxed{1.0825}$

$x_2 = 1.981090061 \approx \boxed{1.9811}$

(4 decimal places)

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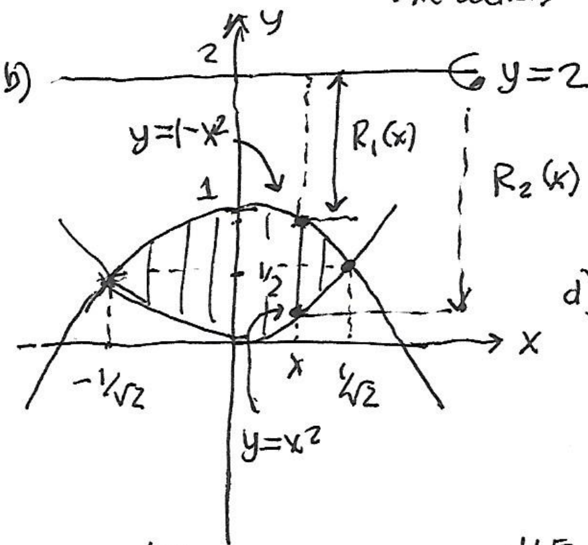
② a) $y=x^2$, $y=1-x^2$ $\xrightarrow[\text{intersection}]{}$ $x^2=1-x^2 \to 2x^2=1$, $x^2=1/2$, $x=\pm 1/\sqrt{2}$

$\hookrightarrow y=(\pm 1/\sqrt{2})^2=1/2$

b)



c) radii:

larger $R_2(x) = 2-x^2$

smaller $R_1(x) = 2-(1-x^2) = 1+x^2$

d) $A(x)=\pi\left(R_2(x)^2-R_1(x)^2\right)$

$=\pi\left((2-x^2)^2-(1+x^2)^2\right)$

$=\pi\left(4-4x^2+\cancel{x^4}-(1+2x^2+\cancel{x^4})\right)$

$=\boxed{\pi(3-6x^2) = 3\pi(1-2x^2)}$

e) $V=\int_{-1/\sqrt{2}}^{1/\sqrt{2}} A(x)\,dx = 2\int_0^{1/\sqrt{2}} 3\pi(1-2x^2)\,dx = 6\pi\left(x-\frac{2}{3}x^3\right)\Big|_0^{1/\sqrt{2}}$

even function

$=6\pi\left(\frac{1}{\sqrt{2}}-\frac{2}{3}\left(\frac{1}{\sqrt{2}}\right)^3\right) = 6\pi\left(\frac{1}{\sqrt{2}}-\frac{2}{3}\frac{1}{2\sqrt{2}}\right) = \frac{6\pi}{\sqrt{2}}\left(1-\frac{1}{3}\right) = \frac{6\pi}{\sqrt{2}}\left(\frac{2}{3}\right) = \boxed{2\pi\sqrt{2}}$

≈ 8.885766

(4 decimal places) $\approx \boxed{8.8858}$

③ a) $I=\int_0^{T/2} At\sin\frac{2\pi t}{T}\,dt = \frac{At^2}{4\pi}$ (maple)

$\overset{b)}{=} \int_0^{\pi} A\left(\frac{Tu}{2\pi}\right)(\sin u)\left(\frac{T}{2\pi}\right)du$ (transformed complete.)

$=\boxed{\frac{AT^2}{4\pi^2}\int_0^{\pi} u\sin u\,du}$ (simplify)

b) $u=\frac{2\pi t}{T}$, $du=\frac{2\pi}{T}dt$

$\downarrow$

$t=\frac{Tu}{2\pi}$

$t=0 \to u=0$

$t=\frac{T}{2} \to u=\frac{2\pi}{T}\left(\frac{T}{2}\right)=\pi$

c) (int by parts)

$U=u$, $dV=\sin u\,du$

$dU=du$, $V=\int\sin u\,du=-\cos u$

(capital letters to avoid confusion)

$\int u\sin u\,du = u(-\cos u)-\int(-\cos u)\,du$

$= -u\cos u+\underbrace{\int\cos u\,du}_{\sin u\ +\ C}$

so $I=\frac{AT^2}{4\pi^2}\underbrace{(\sin u-u\cos u)\Big|_0^{\pi}}_{\sin\pi-\pi\underbrace{\cos\pi}_{-1}=\pi;\ -0+0} = \frac{AT^2}{4\pi^2}\cdot\pi = \boxed{\frac{AT^2}{4\pi}}$

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④ a) $v(r) = \frac{PR^2}{4\eta\ell} \underbrace{\cos\left(\frac{\pi r}{2R}\right)}$

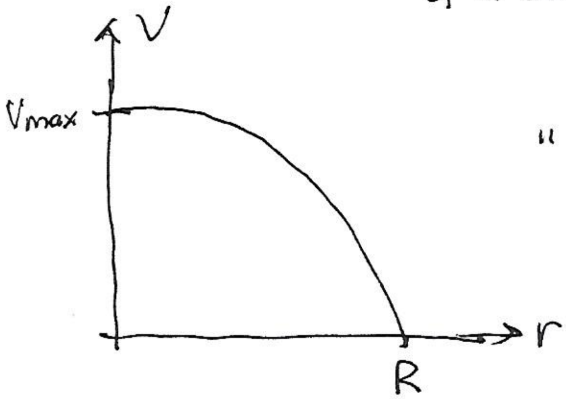
max value at $r=0$:

$\cos \to 1$

so $v(0) = \boxed{\frac{PR^2}{4\eta\ell} = V_{max}}$

b) $v(R) = \frac{PR^2}{4\eta\ell}\cos 0 = 0$

so it decreases from V_{max} to 0 in the shape of a cosine graph.



"generic graph"

c) $V_{avg} = \frac{1}{R}\int_0^R v(r)\,dr = \frac{1}{R}\int_0^R \frac{PR^2}{4\eta\ell}\cos\left(\frac{\pi r}{2R}\right) dr$

$= \frac{PR}{4\eta\ell}\int_0^R \cos\left(\underbrace{\frac{\pi r}{2R}}\right) dr$

$u = \frac{\pi r}{2R} \to du = \frac{\pi}{2R}\,dr$

$dr = \frac{du}{(\pi/2R)}$

$r=0 \to u=0$

$r=R \to u=\frac{\pi}{2}$

$= \frac{PR}{4\eta\ell}\int_0^{\pi/2} \cos\left(\underbrace{\frac{\pi r}{2R}}_{u}\right)\left(\frac{du}{\pi/2R}\right)$

$= \frac{PR}{4\eta\ell}\left(\int_0^{\pi/2}\cos u\,du\right)\left(\frac{2R}{\pi}\right)$

$= \frac{2PR^2}{\pi\, 4\eta\ell}\underbrace{\int_0^{\pi/2}\cos u\,du}_{\sin u\,\Big|_0^{\pi/2} = 1-0=1}$ (simplify)

$= \frac{2}{\pi}\left(\frac{PR^2}{4\eta\ell}\right)$

$= \frac{2}{\pi}V_{max}$

so $\boxed{\frac{V_{avg}}{V_{max}} = \frac{2}{\pi}} \approx 0.6366$

$\approx \boxed{0.64}$

2 decimal places

so we get: (interpretation)