MAT1505-02/03 23F Test 1 Answers

(1) a) $\int x^3 e^{-\frac{x^2}{a^2}}\,dx = \int x^2 e^{-x^2/a^2}\, \underbrace{x\,dx} = \int (-a^2 w)e^{w}\left(-\frac{a^2}{2}dw\right)$

$\left[w = -\frac{x^2}{a^2} \rightarrow x^2 = -a^2 w \right]$

$dw = -\frac{2x\,dx}{a^2}$

$x\,dx = -\frac{a^2}{2}dw$

$= \frac{a^4}{2}\int \underbrace{w}_{u}\underbrace{e^w dw}_{dv}$

$\left[du = dw \leftarrow v = \int e^w dw = e^w \right]$

$= \frac{a^4}{2}\left[we^w - \underbrace{\int e^w dw}_{e^w} \right]$

$= \frac{a^4}{2}(w-1)e^w + C.$

$= \frac{a^4}{2}\left(-\frac{x^2}{a^2}-1\right)e^{-x^2/a^2} + C$

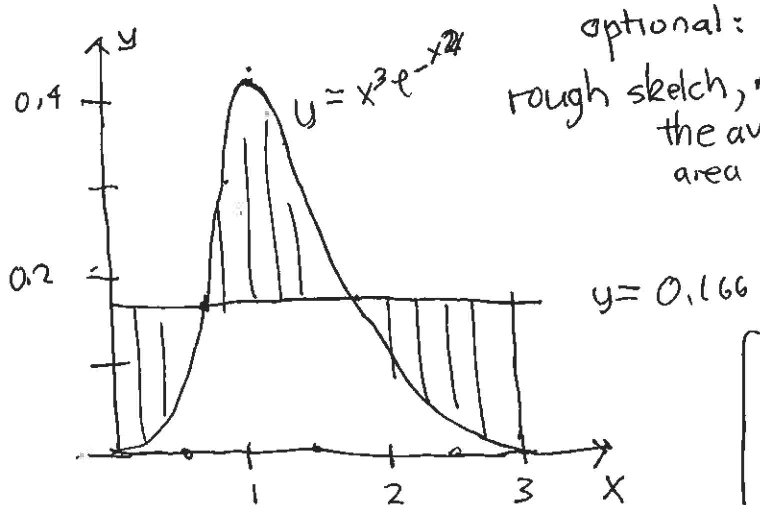
$= \boxed{-\frac{a^4}{2}\left(1+\frac{x^2}{a^2}\right)e^{-\frac{x^2}{a^2}} + C}$

b) $f_{avg} = \frac{1}{3a}\int_0^{3a} x^3 e^{-x^2/a^2}\,dx$

$= \frac{1}{3a}\left[-\frac{a^4}{2}\left(1+\frac{x^2}{a^2}\right)e^{-x^2/a^2} \right]\Bigg|_0^{3a}$

$= \frac{a^3}{6}\left[-\underbrace{(1+\frac{9a^2}{a^2})}_{10}e^{-9a^2/a^2} + \underbrace{(1+0)e^0}_{1} \right]$

$= \boxed{\frac{a^3}{6}(1-10e^{-9})} \approx 0.16646 \approx \boxed{0.166}$



optional:
rough sketch, middle hump poorly drawn but the average line does divide up the area above and below the average line on my Maple plot.

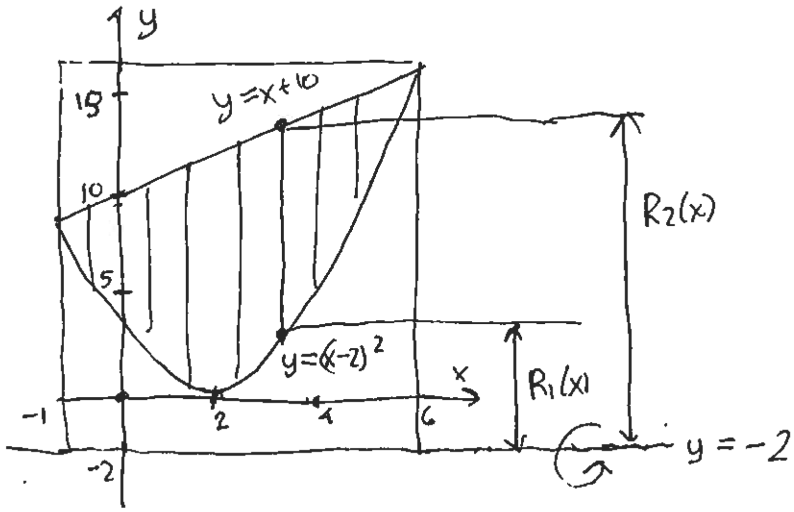
[Note $10e^{-9} \approx 0.0012 \ll 1$ is very small compared to 1 so makes little difference in this formula]

MAT1505-02/03 23F Test 1 Answers (2)

② a) $y=(x-2)^2$, $y=x+10$ $\longrightarrow$ $(x-2)^2=x+10$ (intersection pts)

$$x^2-4x+4=x+10$$
$$x^2-5x-6=0$$
$$x=\frac{5\pm\sqrt{25-4(-6)}}{2}=\frac{5\pm\sqrt{49}}{2}=\frac{5\pm 7}{2}=-1,6$$

b)



$$R_1(x)=(x-2)^2-(-2)=x^2-4x+4+2=x^2-4x+6$$

$$R_2(x)=x+10-(-2)=x+12$$

c) $V=\pi\int_{-1}^{6} R_2(x)^2-R_1(x)^2\,dx=\pi\int_{-1}^{6}\underbrace{(x+12)^2-(x^2-4x+6)^2}_{=-x^4+8x^3-27x^2+72x+108 \text{ Maple}}\,dx$

$=\boxed{\frac{5488}{5}\pi}\approx 3448.2\approx\boxed{3448}$ Maple

③ a) $G=2\int_0^1 x-(1-(1-x)^q)\,dx=2\int_0^1 \underbrace{x-1}_{\frac{x^2}{2}-x}+\overbrace{(1-x)^q\,dx}^{u^q(-du)}$

$u=1-x$
$du=-dx$

$$=2\left[\frac{x^2}{2}-x-\frac{(1-x)^{q+1}}{q+1}\right]\Big|_0^1$$
$$=\left[x^2-2x-\frac{2}{q+1}(1-x)^{q+1}\right]\Big|_0^1$$
$$=1-2-0-\left(0-\frac{2}{q+1}\right)=\frac{2}{q+1}-1$$
$$=\frac{2-(q+1)}{q+1}=\boxed{\frac{1-q}{1+q}}$$

$$\int(1-x)^q\,dx=-\int u^q\,du=-\frac{u^{q+1}}{q+1}+C=-\frac{(1-x)^{q+1}}{q+1}+C$$

b) since $q\geq 0$, the numerator is smaller than the denominator as long as $q<1$ which it is so this is a positive fraction less than 1.

Calculus: Basic Functions for instant recall

	diff	int	simple u-sub
power	$\frac{d}{dx} x^n = n x^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$	$\int (ax+b)^n\,dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + C$
ln	$\frac{d}{dx} \ln\lvert x\rvert = \frac{1}{x}$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$	$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln\lvert ax+b\rvert + C$
exp	$\frac{d}{dx} e^x = e^x$	$\int e^x\,dx = e^x + C$	$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$
trig	$\frac{d}{dx}\sin x = \cos x$	$\int \cos x\,dx = \sin x + C$	$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$
	$\frac{d}{dx}\cos x = -\sin x$	$\int \sin x\,dx = -\cos x + C$	$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$

chain/ u-sub $\quad \frac{d}{dx} f(u) = \underbrace{f'(u)}_{\frac{df}{du}} \frac{du}{dx}$ $\qquad \int f(u(x))\frac{du(x)}{dx}\,dx = \int f(u)\,du = F(u) + C = F(u(x)) + C$

if $F'(x) = f(x)$ "antiderivative"

additive constant $\quad \frac{d}{dx}(f(x) + c) = \frac{d}{dx} f(x)$

multiplicative constant $\quad \frac{d}{dx}(c f(x)) = c\frac{d}{dx} f(x)$ $\qquad \int c\, f(x)\,dx = c\int f(x)\,dx$ $\quad \Big]\ + \neq *\ !$

You are expected to be able to do any of the above explicit integrals by hand, or any that can be reduced to them by an obvious u-substitution. Any derivative or integral you are uncertain of you are expected to check symbolically with Maple or your graphing calculator. There is no excuse for getting a derivative or integral wrong with technology at your fingertips.

MAT 2500 = CALC 3 and MAT2705 = DEwLinAlg assume these basic operations from CALC 1 and CALC 2 and build on them.

Regions of the plane : relationships between variables

Exercise. $y=x^4$, $y=x^{1/3}$ enclose a region R of the plane.

a) Find the area of R.

b) Form a solid by rotating R around the axis $y=2$. Find its volume.

c) Form a solid by rotating R around the axis $x=-1/2$. Find its volume.

Do we really care about answering such artificial questions? Of course not. This is practice in understanding relationships between variables which is important in some real calculus applications.

intersection pts

$x^4 = x^{1/3}$

$(x^4)^3 = (x^{1/3})^3$

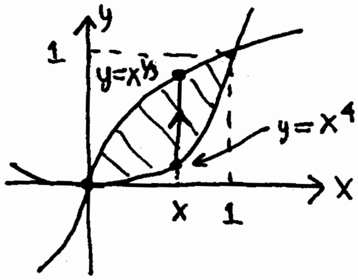
$x^{12} = x$

$x^{12} - x = 0$

$x(x^{11}-1) = 0$

$x=0, \quad x=1$

$y=0 \quad\quad y=1$

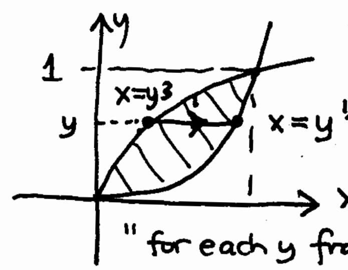


"for each x from 0 to 1, y goes from x^4 to $x^{1/3}$"

OR CHANGE OF INDEPENDENT VARIABLE:

$y=x^{1/3} \to y^3 = (x^{1/3})^3 = x \to x = y^3$

$y = x^4 \to y^{1/4} = (x^4)^{1/4} = \underbrace{|x| = x}_{x>0 \text{ here.}} \to x = y^{1/4}$



"for each y from 0 to 1, x goes from y^3 to $y^{1/4}$"

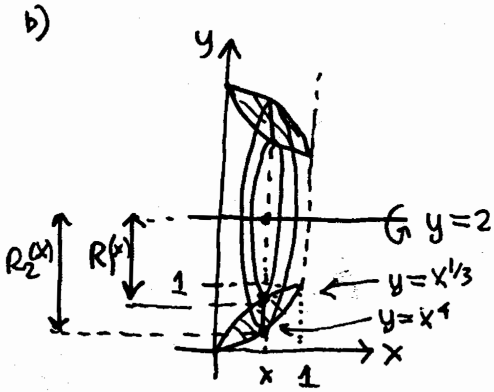
a) x-approach: $L(x) = x^{1/3} - x$ (upper−lower)

$A = \int_0^1 L(x)\,dx = \int_0^1 x^{1/3} - x\,dx$

y-approach: $L(y) = y^{1/4} - y^3$ (right −left)

$A = \int_0^1 L(y)\,dy = \int_0^1 y^{1/4} - y^3\,dy$.

b)



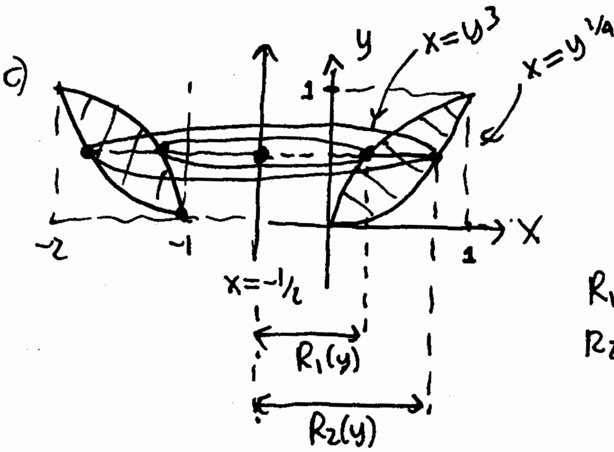
$A(x) = \pi R_2(x)^2 - \pi R_1(x)^2 = \pi(R_2(x)^2 - R_1(x)^2)$

$V = \int_0^1 A(x)\,dx = \int_0^1 \pi\left((2-x^{1/3})^2 - (2-x^4)^2\right)dx$

$R_1 = 2 - x^{1/3}$

$R_2 = 2 - x^4$

c)



$A(y) = \pi(R_2(y))^2 - \pi R_1(y)^2 = \pi(R_2(y)^2 - R_1(y)^2)$

$V = \int_0^1 A(y)\,dy = \int_0^1 \pi\left((y^{1/4}+\tfrac{1}{2})^2 - (y^3+\tfrac{1}{2})^2\right)dy$

$R_1(y) = y^3 - (-\tfrac{1}{2}) = y^3 + \tfrac{1}{2}$

$R_2(y) = y^{1/4} - (-\tfrac{1}{2}) = y^{1/4} + \tfrac{1}{2}$

$\boxed{5.5a}$ changing the variable in an integral

Example $\int 2x\sqrt{1+x^2}\,dx = \int \underbrace{(1+x^2)}_{}^{1/2}\, \underbrace{2x\,dx}_{}$

obvious composition

$u = 1+x^2$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$ — du!

$$= \int u^{1/2}\,du$$

$$= \frac{u^{3/2}}{3/2} + c = \frac{2}{3}u^{3/2} + c \quad \underset{\text{go back}}{=\!=} \quad \frac{2}{3}(1+x^2)^{3/2} + C$$

EASY! But the integrand does not have to be precisely of this chain rule form.

Example $\int x^3\sqrt{1+x^2}\,dx = \int \underbrace{(1+x^2)}_{}^{1/2}\, x^2\, \underbrace{x\,dx}_{\frac{du}{2}}$

$u = 1+x^2$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

leftover factor, just re-express in terms of $u = 1+x^2$

$u-1 = x^2$

$x^2 = u-1$!

$$= \int u^{1/2}(u-1)\frac{du}{2} \quad \text{expand out}$$

$$= \frac{1}{2}\int u^{1/2}(u-1)\,du$$

$$= \frac{1}{2}\int u^{1/2}u - u^{1/2}\,du = \frac{1}{2}\int u^{3/2} - u^{1/2}\,du$$

$$= \frac{1}{2}\left(\frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2}\right) + c = \frac{1}{2}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + c$$

$$= \frac{1}{5}u^{5/2} - \frac{1}{3}u^{3/2} + c \quad \text{(simplified in new variable)}$$

$$= \frac{1}{5}(1+x^2)^{5/2} - \frac{1}{3}(1+x^2)^{3/2} + C \quad \text{(return to original variable)}$$

changing the variable in an integral

What about definite integrals? The only extra piece is adding the limits of integration.

$$\int x^3\sqrt{1+x^2}\,dx = \int u^{3/2}-u^{1/2}\,du =$$
$$= \frac{1}{5}u^{5/2}-\frac{1}{3}u^{3/2}+C$$
$$= \frac{1}{5}(+x^2)^{5/2}-\frac{1}{3}(1+x^2)^{3/2}+C$$

But $\displaystyle\int_0^1 x^3\sqrt{1+x^2}\,d\underset{\sim}{x} = \int_{x=0}^{x=1} x^3\sqrt{1+x^2}\,dx = \int_{u=?}^{u=?} u^{3/2}-u^{1/2}\,du$

identifies variable whose values are limits → more explicit

here we need u values

Calculate corresponding values:

$$u=1+x^2 \begin{cases} x=0 \rightarrow u=1+0^2=1 & \text{(lower)} \\ x=1 \rightarrow u=1+1^2=2 & \text{(upper)} \end{cases}$$

$$\boxed{=\int_1^2 u^{3/2}-u^{1/2}\,du}$$

new form of definite integral

We have 2 choices.

1) go back to original variable
2) stay with new variable

$$\int_0^1 x^3\sqrt{1+x^2}\,dx = \frac{1}{5}(1+x^2)^{5/2}-\frac{1}{3}(1+x^2)^{3/2}\Big|_0^1$$
$$= \frac{1}{5}(2)^{5/2}-\frac{1}{3}(2)^{3/2}-\left[\frac{1}{5}(1)^{5/2}-\frac{1}{3}(1)^{3/2}\right]$$

$$= \frac{1}{5}u^{5/2}-\frac{1}{3}u^{3/2}\Big|_1^2 = \frac{1}{5}(2)^{5/2}-\frac{1}{3}(2)^{3/2}-\left[\frac{1}{5}(1)^{5/2}-\frac{1}{3}(1)^{3/2}\right]$$

same

just different order of operations

finish evaluation
$$= 2^{3/2}\left(\frac{2}{5}-\frac{1}{3}\right)-\left(\frac{1}{5}-\frac{1}{3}\right)$$
$$= \frac{2^{3/2}}{15}+\frac{2}{15} = \boxed{\frac{2}{15}(\sqrt{2}+1)}$$

always simplify, check with Maple!