

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, IDENTIFYING expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of each problem. Keep answers EXACT (but give decimal approximations for interpretation when appropriate). Indicate where technology is used and what type (Maple, GC). Only use technology to CHECK hand calculations, not substitute for them.

## pledge

When you have completed the exam, please read and sign the dr bob integrity pledge and hand this test sheet in on top of your answer sheets as a cover page, with the first test page facing up:

"During this examination, all work has been my own. I give my word that I have not resorted to any ethically questionable means of improving my grade or anyone else's on this examination and that I have not discussed this exam with anyone other than my instructor, nor will I until after the exam period is terminated for all participants."

Signature: \_\_\_\_\_

Date: \_\_\_\_\_

1. Consider the function  $f(x) = x^3 e^{-\frac{x^2}{a^2}}$  on the interval  $0 \leq x \leq 3a$  where  $a > 0$  is a constant parameter.

a) Evaluate the integral  $\int f(x) dx$  by first doing the obvious variable substitution  $w = -\frac{x^2}{a^2}$  indicated by the

composed function, then do an integration by parts on the result. [Check your antiderivative is correct with technology!]

b) Evaluate the exact average value of this function on the given interval. What is the 3 decimal place value of this number when  $a = 1$ ?

**Optional. Do this only if you have extra time at the end of the exam.**

c) Set  $a = 1$  in order to plot this function on this interval together with its average value, and make a rough sketch of what you see, completely labeled by tickmarks, axis labels, graph labels, etc. Does your average value line look about right?

2. Consider the region  $R$  the plane between the curves  $y = (x - 2)^2$  and  $y = x + 10$  and evaluate the volume of the solid obtained by rotating that region around the axis  $y = -2$  by following these steps:

a) Find the intersection point values of  $x$ .

b) Sketch a graph illustrating this region  $R$  of the plane shaded by vertical lines, including the axis of rotation (labeled by its equation), and indicating a typical linear cross section of the region at a typical value of  $x$  and labeling and indicating clearly the two relevant radii of the corresponding circles of revolution there needed for evaluating the plane cross-sectional area, as well as, relevant tickmarks etc (like the examples done by bob).

c) Write down the integral for the volume of the solid and simplify the integrand by expanding it and combining like terms (you may use Maple to do this: Simplify).

d) This integrand is just a polynomial function but the numbers are tedious to plug in. Just do this integral exactly using technology and then numerically to the nearest integer.

[turn over!]

3. The Gini index is defined to be twice the area between the Lorenz curve  $y=L(x)$  and the line  $y=x$  on the interval  $0 \leq x \leq 1$ , where  $L(x) \leq x$ , that is, the fractional area under the Lorenz curve since the triangle area

below the line  $y=x$  is  $1/2$ :  $G = \frac{\int_0^1 x - L(x) \, dx}{\frac{1}{2}} = 2 \int_0^1 x - L(x) \, dx$  (so  $G \geq 0$ ). The largest  $G$  can be is 1 when

$L(x) = 0$ , so the Gini index must therefore satisfy  $0 \leq G \leq 1$ .

The Pareto Lorenz curve is  $L(x) = 1 - (1 - x)^q$ ,  $0 < q < 1$ .

a) Evaluate its Gini index as a function of the Pareto parameter  $q$ . Simplify your result to a single fraction.

b) Is the result a positive proper fraction? Why?

[The Gini index a measure of inequality in the distribution of quantities like income and wealth. You can read about it in the textbook. Or not.]

► **solution**