MAT2705-04/05 22S Quiz 9 Print Name (Last, First) ______________________________

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (**but give decimal approximations for interpretation**). Indicate where technology is used and what type (Maple, GC).

1. $y''(t) + 25\,y(t) = \cos(4\,t), y(0) = 0, y'(0) = 0$.

a) Solve this IVP step by step.

b) Express this solution in the form $y = A(t)\,\sin(\omega_-\,t)$ using the trig identity

$\cos(A) - \cos(B) = 2\sin\left(\frac{B-A}{2}\right)\sin\left(\frac{A+B}{2}\right)$., where $A(t)$ is the envelope function for the faster oscillation in this beating problem, with envelope period T_- .

c) Use Maple to plot the solution with the pair of envelope curves for $t = 0..T_-$ showing 2 beats and scan to include in your submitted quiz.

2. $\frac{d^2}{dt^2}\,y(t) + 2\,\frac{d}{dt}\,y(t) + 17\,y(t) = 65\cos(4\,t)$, $y(0) = 0$, $y'(0) = 0$ [Maple notation].

a) State Maple's solution of the initial value problem (use function notation $y(t)$).

b) Put the DE into standard linear form (unit leading coefficient). Then identify the values of the damping constant and characteristic time $k_0 = 1/\tau_0$, the natural frequency ω_0 , and the quality factor $Q = \omega_0\,\tau_0$, exactly and numerically.

c) Find the general solution by hand, showing all steps (including the matrix solution steps).

d) Find the solution satisfying the initial conditions, showing all steps.

e) Express the steady state solution in phase-shifted cosine form. [Make sure you use a diagram to justify your values.]

f) Now find the amplitude $A(\omega)$ of the response function for the driving function $65\cos(4\,t) \to 65\cos(\omega\,t)$ and plot for ω=0..W where W is the nearest integer to $5\,\omega_0$.

g) Use calculus to find the exact frequency and amplitude of the resonance peak, and 3 decimal place decimal values.

▶ solution

① a) $y'' + 5^2 y = \cos(4t)$

$y_h = c_1\cos 5t + c_2\sin 5t$

$y_p = c_3\cos 4t + c_4\sin 4t$

$y_p'' = -16(c_3\cos 4t + c_4\sin 4t)$

$y_p'' + 25y_p = (25-16)c_3\cos 4t + (25-16)c_4\sin 4t = \cos 4t$

$9c_3 = 1 \to c_3 = 1/9$ $\qquad$ $9c_4 = 0 \to c_4 = 0$

$y_p = \frac{1}{9}\cos 4t$

$y = c_1\cos 5t + c_2\sin 5t + \frac{1}{9}\cos 4t$

$y' = -5c_1\sin 5t + 5c_2\cos 5t - \frac{4}{9}\sin 4t$

$y(0) = c_1 + 1/9 = 0 \to c_1 = -1/9$

$y'(0) = 5c_2 = 0 \to c_2 = 0$

$\boxed{y = \frac{1}{9}(\cos 4t - \cos 5t)}$

(b) $= \frac{1}{9}(2)\left(\sin \frac{5-4}{2}t \,\underbrace{\sin \frac{5+4}{2}t}\right)$ $\quad \omega_- \quad \omega_+$

$= \left(\frac{2}{9}\sin\frac{1}{2}t\right)\sin\frac{9}{2}t$

$A(t)$: $T_- = \frac{2\pi}{1/2} = 4\pi$ (longer period)

c) $t = 0..4\pi$ will show 2 beats (see Maple scan)

Note $T_- = \frac{2\pi}{9/2} = \frac{4\pi}{9} = \frac{1}{9}T_+$

so see 9 cycles in the two beats

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$\boxed{\begin{array}{l}\omega=4 \approx \omega_0 = 4.12 \\ \text{a bit less than } \omega_0 \\ \text{so phase shift a bit} \\ \text{less than } 90^\circ\end{array}}$

(2) $y'' + 2y' + 17y = 65\cos 4t$, $y(0) = 0 = y'(0)$

a) Maple: $y = \underbrace{e^{-t}\left(-\cos 4t - \frac{33}{4}\sin 4t\right)}_{y_h \text{ transient}} + \underbrace{(\cos 4t + 8\sin 4t)}_{y_p \text{ steady state}}$

b) $1\,y'' + 2y' + 17y = 65\cos 4t$

$\boxed{\begin{array}{ll} k_0 = 2 & \omega_0 = \sqrt{17} \approx 4.123 \\ \tau_0 = \frac{1}{2} & \longrightarrow Q = \omega_0\tau_0 \approx 2.062 = \frac{\sqrt{17}}{2}\end{array}}$

c) $y_h = e^{rx}$: $\underbrace{(r^2+2r+17)}_{=0} e^{rx} = 0$

$r = \frac{-2 \pm \sqrt{4-4\cdot 17}}{2} = -1 \pm 4i$

$e^{rt} = e^{(-1\pm 4i)t} = e^{-t}e^{\pm 4i} \to \{e^{-t}\cos 4t,\ e^{-t}\sin 4t\}$

$y_h = e^{-t}(c_1\cos 4t + c_2\sin 4t)$

$17\,[\ y_p = c_3\cos 4t + c_4\sin 4t$

$2\,[\ y_p' = -4c_3\sin 4t + 4c_4\cos 4t$

$1\,[\ y_p'' = -16c_3\cos 4t - 16c_4\sin 4t$

$y_p'' + 2y_p' + 17y_p = [(17-16)c_3 + 8c_4]\cos 4t + [-8c_3 + (17-16)c_4]\sin 4t$

$= \underbrace{(c_3 + 8c_4)}_{=65}\cos 4t + \underbrace{(-8c_3 + c_4)}_{=0}\sin 4t = 65\cos 4t$

$\begin{bmatrix}1 & 8\\ -8 & 1\end{bmatrix}\begin{bmatrix}c_3\\ c_4\end{bmatrix} = \begin{bmatrix}65\\ 0\end{bmatrix} \to \begin{bmatrix}c_3\\ c_4\end{bmatrix} = \frac{1}{1+64}\begin{bmatrix}1 & -8\\ 8 & 1\end{bmatrix}\begin{bmatrix}65\\ 0\end{bmatrix} = \frac{65}{65}\begin{bmatrix}1\\ 8\end{bmatrix} = \begin{pmatrix}1\\ 8\end{pmatrix}$

$y_p = \cos 4t + 8\sin 4t$

$\boxed{y = e^{-t}(c_1\cos 4t + c_2\sin 4t) + \cos 4t + 8\sin 4t}$

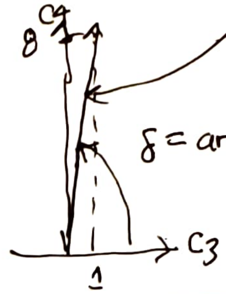
d) $y' = -e^{-t}(c_1\cos 4t + c_2\sin 4t) + e^{-t}(-4c_1\sin 4t + 4c_2\cos 4t) - 4\sin 4t + 32\cos 4t$

$y(0) = c_1 + 1 = 0 \to c_1 = -1$

$y'(0) = -c_1 + 4c_2 + 32 = 0 \to c_2 = \frac{1}{4}(-32 + (-1)) = -\frac{33}{4}$

$\boxed{y = e^{-t}\left(-\cos 4t - \frac{33}{4}\sin 4t\right) + \cos 4t + 8\sin 4t}$

e)



$A = \sqrt{1+64} = \sqrt{65} \approx 8.062$

$\delta = \arctan 8 \approx 83^\circ$

$\boxed{y_p = \sqrt{65}\cos(4t - \arctan 8)}$

$17\,\{\ y_p = c_3\cos\omega t + c_4\sin\omega t$

$2\,[\ y_p' = -\omega c_3\sin\omega t + \omega c_4\cos\omega t$

$1\,[\ y_p'' = -\omega^2 c_3\cos\omega t - \omega^2 c_4\sin\omega t$

$y_p'' + 2y_p' + 17y_p = [(17-\omega^2)c_3 + 2\omega c_4]\cos\omega t + [-2\omega c_3 + (17-\omega^2)c_4]\sin\omega t = 65\cos 4t$

$\begin{bmatrix}17-\omega^2 & 2\omega\\ -2\omega & 17-\omega^2\end{bmatrix}\begin{bmatrix}c_3\\ c_4\end{bmatrix} = \begin{bmatrix}65\\ 0\end{bmatrix}$

$\begin{bmatrix}c_3\\ c_4\end{bmatrix} = \frac{1}{(17-\omega^2)^2 + 4\omega^2}\begin{bmatrix}17-\omega^2 & -2\omega\\ 2\omega & 17-\omega^2\end{bmatrix}\begin{bmatrix}65\\ 0\end{bmatrix}$

$= \frac{65}{(17-\omega^2)^2 + 4\omega^2}\begin{bmatrix}17-\omega^2\\ 2\omega\end{bmatrix}$

$A(\omega) = \sqrt{c_3^2 + c_4^2} = \frac{65}{(\cdots)}\left[(17-\omega^2)^2 + 4\omega^2\right]^{1/2}$

$= \frac{65}{\sqrt{(17-\omega^2)^2 + 4\omega^2}}$

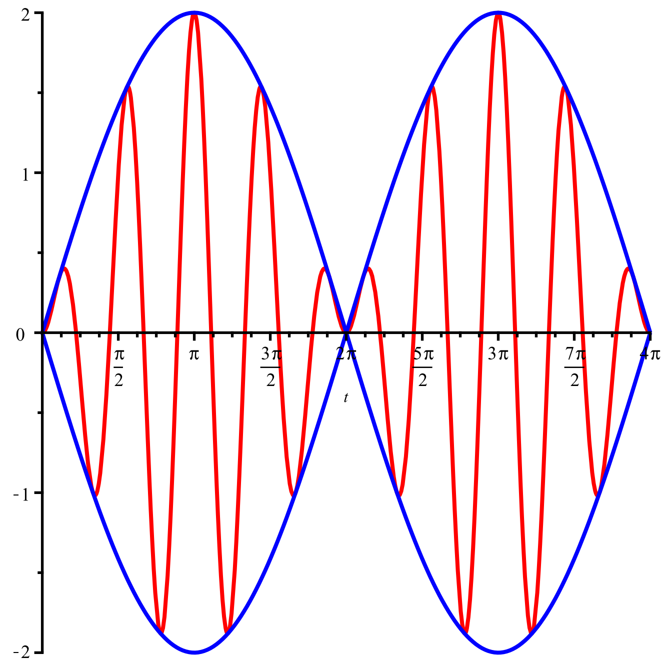
$0 = A'(\omega) = (\cdots)^{-3/2}\,65\left[2(\omega^2-17)(2\omega) + 8\omega\right]$

$= \frac{4\cdot 65\,\omega}{(\cdots)^{3/2}}\left[\omega^2 - 17 + 2\right] \to \omega^2 = 15$

$\boxed{\begin{array}{l}\omega = \sqrt{15} \approx 3.873 \\ A(\sqrt{15}) = \frac{65}{8} = 8.125 \\ \quad\text{Maple}\end{array}}$

$5\omega_0 \approx 20$

> $plot\left(\left[-\cos(5\,t)+\cos(4\,t), \left(2\cdot\sin\left(\frac{1}{2}\,t\right)\right), -\left(2\cdot\sin\left(\frac{1}{2}\,t\right)\right)\right], t=0..\frac{2\pi}{\frac{1}{2}}, color=[red, blue, blue]\right)$



Bonus plot:

> $plot(A(\omega), \omega=0..20, gridlines=true, 0..9)$