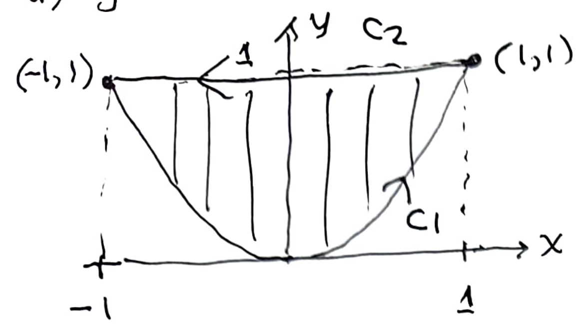
MAT2500-05 22F Final Exam C1

① $y=x^2$, $y=1$ $\rightsquigarrow$ $x^2=1 \to x=\pm 1$ intersection pts

a)



d) $\to$ $\begin{cases} C_1: \ x=t,\ y=t^2,\ t=x=-1..1 \\ \vec{r}_1(t)=\langle t, t^2\rangle,\ \vec{r}_1'(t)=\langle 1, 2t\rangle \\ C_2: \ x=t,\ y=1, \quad t=1..-1 \text{ (reverse)} \\ \vec{r}_2(t)=\langle t, 1\rangle,\ \vec{r}_2'(t)=\langle 1, 0\rangle \end{cases}$

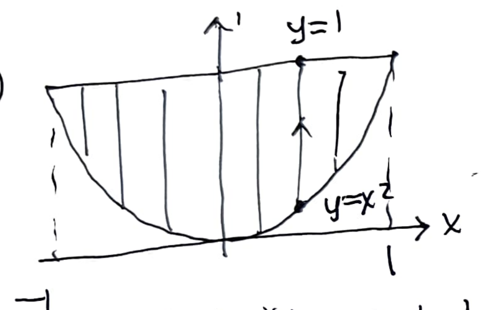
b) $\int_C -x^2y^2\,dx + x^2y\,dy = \int F_1\,dx + F_2\,dy$ $\quad \boxed{\vec{F}=\langle -x^2y^2, x^2y\rangle}$

$F_1=-x^2y^2$, $F_2=x^2y$

$\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial x}(x^2y) - \frac{\partial}{\partial y}(-x^2y^2)$

$= \boxed{2xy+2x^2y} = 2(x+x^2)y$

c)



$y=x^2..1$ while $x=-1..1$

$\iint_R \frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\,dA = \int_{-1}^{1}\int_{x^2}^{1} 2(x+x^2)y\,dy\,dx$

$=\int_{-1}^{1} \cancel{2}(x+x^2)\frac{y^2}{\cancel{2}}\Big|_{y=x^2}^{y=1} dx = \int_{-1}^{1}(x+x^2)(1-\underbrace{(x^2)^2}_{x^4})\,dx$

$=\int_{-1}^{1} x+x^2-x^5-x^6\,dx = \frac{x^2}{2}+\frac{x^3}{3}-\frac{x^6}{6}-\frac{x^7}{7}\Big|_{-1}^{1}$

$=\underbrace{\left[\frac{1}{2}(x^2)-\frac{x^6}{6}\right]\Big|_{-1}^{1}}_{\text{even} \to 0} + \underbrace{\frac{x^3}{3}-\frac{x^7}{7}}_{\text{odd}}\Big|_{-1}^{1} = 2\left(\frac{1}{3}-\frac{1}{7}\right) = 2\frac{(7-3)}{21} = \boxed{\frac{8}{21}}$

$F(x,y)=\langle -x^2y^2, x^2y\rangle$

d) C_1: $\vec{F}(\vec{r}_1(t)) = \langle -t^2(t^2)^2, t^2(t^2)\rangle = \langle -t^6, t^4\rangle$

$\vec{F}(\vec{r}_1(t))\cdot\vec{r}_1'(t) = \langle -t^6, t^4\rangle\cdot\langle 1, 2t\rangle = -t^6+2t^5$

$\int_{C_1}\vec{F}\cdot d\vec{r} = \int_{-1}^{1} -t^6+2t^5\,dt = -\frac{t^7}{7}+\frac{2t^6}{6}\Big|_{-1}^{1} = -\frac{1}{7}(2)+0 = \boxed{-\frac{2}{7}}$

C_2: $\vec{F}(\vec{r}_2(t)) = \langle -t^2\cdot 1^2, t^2\cdot 1\rangle = \langle -t^2, t^2\rangle$

$\vec{F}(\vec{r}_2(t))\cdot\vec{r}_2'(t) = \langle -t^2, t^2\rangle\cdot\langle 1, 0\rangle = -t^2$

$\int_{C_2} F\cdot d\vec{r} = \int_{1}^{-1} -t^2\,dt = -\frac{t^3}{3}\Big|_{1}^{-1} = \frac{t^3}{3}\Big|_{-1}^{1} = \boxed{\frac{2}{3}}$

$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1}\vec{F}\cdot d\vec{r} + \int_{C_2}\vec{F}\cdot d\vec{r} = -\frac{2}{7}+\frac{2}{3} = 2\left(\frac{1}{3}-\frac{1}{7}\right) = \frac{2(7-3)}{21} = \boxed{\frac{8}{21}}$

e) yes!

MAT2500-05 22F Final Exam (2)

② a) $\vec{F} = \langle \underbrace{3x^2yz - 3y}_{F_1},\ \underbrace{x^3z - 3x}_{F_2},\ \underbrace{x^3y + 2z}_{F_3} \rangle$

$$\text{curl } \vec{F} = \left\langle \frac{\partial}{\partial y}F_3 - \frac{\partial}{\partial z}F_2,\ \frac{\partial F_1}{\partial z} - \frac{\partial}{\partial x}F_3,\ \frac{\partial F_2}{\partial x} - \frac{\partial}{\partial y}F_1 \right\rangle$$

$$= \left\langle \frac{\partial}{\partial y}(x^3y+2z) - \frac{\partial}{\partial z}(x^3z-3x),\ \frac{\partial}{\partial z}(3x^2yz-3y) - \frac{\partial}{\partial x}(x^3y+2z),\ \frac{\partial}{\partial x}(x^3z-3x) - \frac{\partial}{\partial y}(3x^2yz-3y) \right\rangle$$

$$= \langle \cancel{x^3} + 0 - \cancel{x^3} + 0,\ \cancel{3x^2y} - 0 - (\cancel{3x^2y} + 0),\ \cancel{3x^2z} - \cancel{3} - (\cancel{3x^2z} - \cancel{3}) \rangle$$

$= \langle 0, 0, 0 \rangle$ ✓

b) $\vec{F} = \vec{\nabla} f$:

$\frac{\partial f}{\partial x} = F_1 = 3x^2yz - 3y \xrightarrow[\text{wrt } x]{\text{integrate}}$ $f = \int 3x^2yz - 3y\, dx$

$= 3\left(\frac{x^3}{3}\right)yz - 3yx + C(y,z)$

$= x^3yz - 3xy + C(y,z)$

$\frac{\partial f}{\partial y} = F_2 = x^3z - 3x$

$\frac{\partial f}{\partial y} = x^3z - 3x + \frac{\partial C}{\partial y}(y,z)$

$\frac{\partial f}{\partial z} = F_3 = x^3y + 2z$

$\cancel{x^3z - 3x} + \frac{\partial C}{\partial y}(y,z) = \cancel{x^3z - 3x}$ 0

$\frac{\partial C}{\partial y}(y,z) = 0$

integrate wrt y

$C(y,z) = C_2(z)$ (ind of y)

so $f = x^3yz - 3xy + C_2(z)$

$\frac{\partial f}{\partial z} = x^3y - 0 + C_2'(z)$

$\cancel{x^3y} + C_2'(z) = \cancel{x^3y} + 2z$

$C_2'(z) = 2z$

integrate wrt z

$C_2(z) = 2\left(\frac{z^2}{2}\right) + k = z^2 + k$

so $\boxed{f = x^3yz - 3xy + z^2 + k}$ for any k.

MAT2500-05 22F Final Exam (3)

② c) $f(xyz) = x^3yz - 3xy + z^2 + \cancel{k}^{\to 0}$ can set $k = 0$
(will cancel out anyway)

$f(1,1,1) = 1\cdot1\cdot1 - 3\cdot1\cdot1 + 1 = 1 - 3 + 1 = -1$

$f(0,0,0) = 0$

$\int_C \vec{F}\cdot d\vec{r} = f(1,1,1) - f(0,0,0) = \boxed{-1}$

d) $\vec{r}(t) = \langle x, y, z\rangle = \langle t, t^2, t^3\rangle$

$\vec{r}\,'(t) = \langle 1, 2t, 3t^2\rangle$

$\vec{F}(xyz) = \langle 3x^2yz - 3y, x^3z - 3, x^3y + 2z\rangle$

$\vec{F}(\vec{r}(t)) = \langle 3t^2(t^2)(t^3) - 3t^2, t^3(t^3) - 3t, t^3(t^2) + 2t^3\rangle$

$= \langle 3t^7 - 3t^2, t^6 - 3t^2, t^5 + 2t^3\rangle$

$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = (3t^7 - 3t^2)(1) + (t^6 - 3t)(2t) + (t^5 + 2t^3)(3t^2)$

$= 3t^7 - 3t^2 + 2t^7 - 6t^2 + 3t^7 + 6t^5$

$8t^7 - 9t^2 + 6t^5$

$\int_0^1 \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = \int_0^1 8t^7 + 6t^5 - 9t^2\,dt$

$= \frac{\cancel{8}t^8}{\cancel{8}} + \frac{\cancel{6}t^6}{\cancel{6}} - \frac{9t^3}{3}\Big|_0^1 = 1 + 1 - 3 = \boxed{-1}$ ✓ agrees!

MAT2500-05 22F Final Exam (4)

③ a) $\vec{F} = \langle F_1, F_2 \rangle = \left\langle \frac{kx}{x^2+y^2}, \frac{ky}{x^2+y^2} \right\rangle$

$$\frac{\partial F_1}{\partial x} = \frac{\partial}{\partial x}\left(\frac{kx}{x^2+y^2}\right) = k\cdot \frac{(x^2+y^2)(1) - x(2x+0)}{(x^2+y^2)^2}$$

$$= k\,\frac{(x^2+y^2-2x^2)}{(x^2+y^2)^2} = \frac{k(-x^2+y^2}{(x^2+y^2)^2}$$

$$\frac{\partial F_2}{\partial y} = \frac{\partial}{\partial y}\left(\frac{ky}{x^2+y^2}\right) \underset{\text{switch } x \text{ and } y}{=} \cdots = \frac{k(x^2-y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = k\,\frac{((-x^2+y^2)+(x^2-y^2))}{(x^2+y^2)^2} = 0\ !$$

b) $f(x,y) = \frac{1}{2}k\ln(x^2+y^2)$

$$\frac{\partial f}{\partial x} = \frac{1}{2}k\,\frac{1}{x^2+y^2}(2x) = \frac{kx}{x^2+y^2} = F_1 \quad ✓$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}k\,\frac{1}{x^2+y^2}(2y^2) = \frac{ky}{x^2+y^2} = F_2 \quad ✓$$

$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}\left(ky(x^2+y^2)^{-1}\right)$$

$$= -\frac{ky(2x)}{(x^2+y^2)^{+2}}$$

$$= \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}\left(kx(x^2+y^2)^{-1}\right)$$

$$= -\frac{kx(2y)}{(x^2+y^2)^2}$$

$$= -\frac{2kxy}{(x^2+y^2)^2}$$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{-2xy+2xy}{(x^2+y^2)^2} = 0$$

(so admits a potential function)