MAT2500-05 22F Test 2

① a) $f(x,y,z) = x^2 + 5y^2 + 6yz + 5z^2 - 4$

$\vec{\nabla} f(x,y,z) = \boxed{\langle 2x, 10y + 6z, 6y + 10z \rangle}$

$\vec{\nabla} f(2,1,1) = \langle 2(2), 10(1) + 6(1), 6 \cdot 1^2 + 10(1) \rangle = \boxed{\langle 4, 16, 16 \rangle} = 4\langle 1,4,4 \rangle$

$|\vec{\nabla} f(2,1,1)| = 4\, |\langle 1,4,4 \rangle| = 4\sqrt{1+16+16} = \boxed{4\sqrt{33}}$

factor of 4 ignorable for unit direction but not $\vec{\nabla} f(2,1,1)$ itself!

$\hat{u} = \widehat{\vec{\nabla} f(2,1,1)} = \widehat{\langle 1,4,4 \rangle} = \boxed{\dfrac{\langle 1,4,4 \rangle}{\sqrt{33}}}$

b) $\vec{v} = -\vec{OP} = -\langle 2,1,1 \rangle \rightarrow \hat{v} = -\dfrac{\langle 2,1,1 \rangle}{\sqrt{4+1+1}} = -\dfrac{\langle 2,1,1 \rangle}{\sqrt{6}}$

$D_{\hat{v}} f(2,1,1) = \hat{v} \cdot \vec{\nabla} f(2,1,1) = -\dfrac{\langle 2,1,1 \rangle}{\sqrt{6}} \cdot 4\langle 1,4,4 \rangle = -\dfrac{4}{\sqrt{6}}(2+4+4) = -\dfrac{4 \cdot 10}{\sqrt{6}} = \boxed{-\dfrac{40}{\sqrt{6}}}$ or $= -\dfrac{20}{3}\sqrt{6}$

$D_{\hat{v}} f(2,1,1) < 0$ so $\boxed{\text{decreasing}}$ in that direction,

c) $f(2,1,1) = 2^2 + 5 \cdot 1^2 + 6 \cdot 1 \cdot 1 + 5 \cdot 1^2 - 4 = 4 + 5 + 6 + 5 - 4 = 16$

so $f(x,y,z) = 16$: $\boxed{x^2 + 5y^2 + 6yz + 5z^2 - 4 = 16}$

level surface $\neq$ tangent plane

goes thru point (2,1,1)

d) $\vec{r}_0 = \langle 2,1,1 \rangle$, $\vec{a} \propto \vec{\nabla} f(2,1,1) \rightarrow \vec{a} = \langle 1,4,4 \rangle$ [or $\langle 4,16,16 \rangle$]

$\vec{r} = \vec{r}_0 + t\vec{a} = \langle 2,1,1 \rangle + t\langle 1,4,4 \rangle = \boxed{\langle 2+t, 1+4t, 1+4t \rangle = \langle x,y,z \rangle}$

$0 = z = 1 + 4t \rightarrow t = -1/4 \rightarrow x = 2 + (-\tfrac{1}{4}) = \tfrac{7}{4}$, $y = 1 + 4(-\tfrac{1}{4}) = 0$

so intersects xy plane at $\boxed{(x,y) = (7/4, 0)}$ or $(x,y,z) = (\tfrac{7}{4}, 0, 0)$

e) $\boxed{L(x,y,z) = f(2,1,1) + f_x(2,1,1)(x-2) + f_y(2,1,1)(y-1) + f_z(2,1,1)(z-1)}$

$\boxed{= 16 + 4(x-2) + 16(y-1) + 16(z-1)}$ $= 4x + 16y + 16z \underbrace{-8-16}_{-24} \cancel{-16} \cancel{+16}_{0}$

$L(2.1, 0.9, 1.1) = 16 + 4(2.1-2) + 16(.9-1) + 16(1.1-1)$

$= 16 + \underbrace{4(0.1) - 16(0.1) + 16(0.1)}_{0.4 \, \cancel{-1.6} \, \cancel{+1.6} \, 0} = \boxed{16.4}$

small change compared to $f(2,1,1) = 16$

MAT2500-05 22F Test 2 (2)

② a) $f(x,y) = x^3 - 6xy + 8y^3$

$f_x(x,y) = 3x^2 - 6y = 3(x^2 - 2y) = 0 \rightarrow y = x^2/2$

$f_y(x,y) = -6x + 24y^2 = 6(-x + 4y^2) = 0 \rightarrow -x + \cancel{4}\left(\frac{x^2}{\cancel{2}}\right)^2 = 0 \rightarrow -x + x^4 = 0$

$x(x^3 - 1) = 0$

$x = 0$ or $x = 1$.

$y = \frac{0^2}{2} = 0 \quad y = \frac{(1)^2}{2} = \frac{1}{2}$

(not requested)

← was meant to save you time!

"just plug in" to eqns to verify they are solutions!

verify these are critical pts:

$f_x(0,0) = 3 \cdot 0 - 6 \cdot 0 = 0$ ✓

$f_y(0,0) = -6 \cdot 0 + 24 \cdot 0 = 0$ ✓

$f_x(1, \frac{1}{2}) = 3 \cdot 1^2 - 6(\frac{1}{2}) = 3 - 3 = 0$ ✓

$f_y(1, \frac{1}{2}) = -6 \cdot 1 + 24(\frac{1}{2})^2 = -6 + 6 = 0$ ✓

$f(0,0) = \boxed{0}$

$f(1, 1/2) = 1^3 - 6 \cdot 1 \cdot \frac{1}{2} + 8 \cdot (\frac{1}{2})^3$

$= 1 - 3 + \frac{8}{8} = 2 - 3 = \boxed{-1}$

b) $f_{xx} = \frac{\partial}{\partial x}(3x^2 - 6y) = 6x$

$f_{xy} = \frac{\partial}{\partial y}(3x^2 - 6y) = -6$

$f_{yy} = \frac{\partial}{\partial y}(-6x + 24y^2) = 48y$

	(0,0)	(1, ½)
f_{xx}	0	$6 \cdot 1 = 6 > 0$
f_{yy}	0	$48(\frac{1}{2}) = 12 > 0$
f_{xy}	−6	−6
$f_{xx}f_{yy} - f_{xy}^2$	$0 - (-6)^2 = -36 < 0$	$6 \cdot 12 - (-6)^2 = 2 \cdot 36 - 36 = 36 > 0$

saddle point

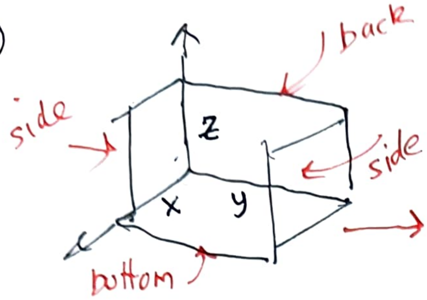
local min?

yes confirms local min.

thought process documented:

not regurgitated conditions listed in textbook

(3)



$x>0,\ y>0,\ z>0$

$V = xyz = 2$ (so none can equal zero!)

$A = \underbrace{xy}_{\text{bot}} + \underbrace{zy}_{\text{back}} + \underbrace{2 \cdot xz}_{\text{sides}}$ extremize!

If you exchanged x & y, it would be enough to identify them in the diagram

eliminate z:

$xyz = 2 \rightarrow z = \frac{2}{xy} \longrightarrow A(x,y) = xy + \left(\frac{2}{xy}\right)(y+2x)$

$= xy + 2\left(\frac{1}{x} + \frac{2}{y}\right) = \boxed{xy + 2(x^{-1} + 2y^{-1})}$

$\boxed{\text{for } x>0,\ y>0}$

$A_x = y + 2(-x^{-2}) = y - \frac{2}{x^2} = 0 \longrightarrow y = \frac{2}{x^2} \rightarrow \frac{1}{y} = \frac{x^2}{2}$

$A_y = x + 2 \cdot 2(-y^{-2}) = x - \frac{4}{y^2} = 0$

$x - 4\left(\frac{x^2}{2}\right)^2 = 0$

$x - x^4 = 0$

$x(1-x^3) = 0$

$\cancel{x=0}$ or $x = 1$ ✓ $\rightarrow y = \frac{2}{1^2} = 2,\ z = \frac{2}{1 \cdot 2} = 1$

standard technique: solve one eqn for one variable, substitute for it in second eqn & solve for second variable, then back sub to evaluate first variable

$(x,y,z) = (1,2,1)$

$A(1,2) = 1 \cdot 2 + 2\left(\frac{1}{1} + \frac{2}{2}\right) = 2 + 2(1+1) = 6$

The open box with smallest area has side lengths 1, back ~~length~~ width 2 and height 1 with area 6.