MAT1505-05 22F Quiz 9 Print Name (Last, First) ________________, ________________

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, IDENTIFYING expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation when appropriate). Indicate where technology is used and what type (Maple, GC). INDICATE where technology is used and what type (Maple, GC). Only use technology to CHECK hand calculations, not substitute for them.

1.a) From the derivative formula $\frac{d}{dx}(\operatorname{arctanh}(x)) = \frac{1}{1-x^2}$, let $f(x) = \operatorname{arctanh}(x)$ and use the taylor series formula to evaluate by differentiation the third degree Taylor polynomial $T_3(x) = \sum_{n=0}^{3}\left(\frac{f^{(n)}(0)\,x^n}{n!}\right)$. Does this agree with the Maple taylor command result?

b) From the fundamental theorem of calculus, since $\operatorname{arctanh}(0) = 0$, we can define $\operatorname{arctanh}(x) = \int_0^x \frac{1}{1-t^2}\,dt$.

Use this formula and the summation formula for a geometric series to evaluate the complete Taylor series $\sum_{n=0}^{\infty}(c_n x^n)$ for this function. Confirm that this agrees with part a) for the first few nonzero terms. What is its radius of convergence (why?)?

c) Evaluate the (positive) fractional error $E_3 = \frac{\arctan(0.2) - T_3(0.2)}{\arctan(0.2)}$ and convert to a percentage. [Since this is a positive series, we cannot use the alternating series estimate for the maximum error to understand how many terms we need to get a certain precision.]

▶ **solution**

a) $f(x) = \operatorname{arctanh}(x)$

$f'(x) = \frac{1}{1-x^2} = (1-x^2)^{-1}$

$f''(x) = -(1-x^2)^{-2}(-2x)$

$= \frac{2x}{(1-x^2)^2}$

$f^{(3)}(x) = \frac{(1-x^2)^2(2) - (2x)(2)(1-x^2)(-2x)}{(1-x^2)^4}$

$= \frac{2(1-x^2)}{(1-x^2)^4}(1-x^2+4x^2) = \frac{2(1+3x^2)}{(1-x^2)^3}$

$f(0) = \operatorname{arctanh}(0) = 0$ $\left(\tanh(0) = \frac{e^0 - e^{-0}}{e^0+e^{-0}} = \frac{1-1}{1+1} = 0\right.$ so $\operatorname{arctanh}(0) = 0$ or use Maple or see part b) $\Big)$

$f'(0) = (1-0)^{-1} = 1$

$f''(0) = \frac{2(0)}{1} = 0$

$f^{(3)}(0) = \frac{2(1+0)}{1} = 2$

$T_3(x) = f(0) + f'(0)\,x + \frac{f''(0)x^2}{2!} + \frac{f^{(3)}(0)x^3}{3!}$

$= 0 + 1\cdot x + 0\cdot x^2 + \frac{2}{6}x^3$

$= \boxed{x + \frac{1}{3}x^3}$

Maple agrees

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b) $\arctan h\, x = \int_0^x \underbrace{\frac{1}{1-t^2}}\, dt = \int_0^x \left(\sum_{n=0}^{\infty} t^{2n}\right) dt = \sum_{n=0}^{\infty} \int_0^x t^{2n}\, dt$

$r = t^2: \sum_{n=0}^{\infty} (t^2)^n$

$= \sum_{n=0}^{\infty} \left. \frac{t^{2n+1}}{2n+1} \right|_0^x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} = \underbrace{\frac{x^1}{1} + \frac{x^3}{3} + \frac{x^5}{5}}_{\text{agrees.}} + \cdots$

$|r| = |t^2| < 1$ for geometric series formula to be valid

$\hookrightarrow |t| < 1 \rightarrow \boxed{|x| < 1}$ $\boxed{\text{radius of convergence} = 1}$

c) $E_3 = \dfrac{\arctan h .2 - \left(.2 + \frac{(.2)^3}{3}\right)}{\arctan h .2} = \dfrac{0.202732 - 0.202667}{0.202732}$

$= \dfrac{0.000066}{} = \boxed{0.00032}$

$\rightarrow \sim \boxed{0.03\%}$ not bad

(4 decimal place accuracy
0.2027)