MAT1505-05 22F Quiz 7 Print Name (Last, First) ______________, ______________

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, IDENTIFYING expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation when appropriate). Indicate where technology is used and what type (Maple, GC). INDICATE where technology is used and what type (Maple, GC). Only use technology to CHECK hand calculations, not substitute for them.

1. The function $f(x) = x\,e^{-kx} \geq 0, \quad 0 \leq x < \infty, \quad k > 0$
can be made into a probability distribution $p(x)$ by dividing by its integral over this semi-infinite interval. You may assume that any polynomial times the exponential has a limit of zero at infinity in evaluating improper integral limits. To check improper integrals using Maple directly you must: > *assume*$(k > 0)$.

a) Evaluate its integral (using Maple for the antiderivative and evaluating the improper integral limit) and state $p(x)$.

b) Evaluate the mean value μ for this probability distribution as a function of k and re-express the distribution in terms of it.

Now proceed with the "standard" such distribution with unit mean setting $\mu = 1$ for the rest of this problem, equivalent to measuring x in units of the mean.

c) Find the value of $x = x_{max}$ where the peak occurs exactly using calculus.

d) Evaluate the probability that the variable x assumes a value less than the peak value.

e) Evaluate the median $x = m$ by using Maple to solve the equation $\int_0^m p(x)\,dx = \frac{1}{2}$ numericaly and give the value to 4 decimal places.

f) Illustrate this by making a rough sketch for the standard distribution and marking off x_{max}, μ, m on the graph by vertical lines from the horizontal axis up to the graph of $p(x)$. Label each such value by its decimal value at the corresponding location on the horizontal axis.

g) **optional.**

Evaluate the standard deviation $\langle (x-\mu)^2 \rangle$ for the standard probability distribution $(\mu = 1)$ numerically and mark off $[1-\sigma, 1+\sigma]$ on the same axis to indicate the values within one standard deviation of the mean.

▶ solution

a) $\int_0^\infty x e^{-kx}\,dx = \lim_{t\to\infty} -\frac{1}{k^2}(kx+1)e^{-kx}\Big|_0^t = \lim_{t\to\infty}\left[\frac{1}{k^2}(0+1)e^0 - \frac{1}{k^2}\underbrace{\frac{kt+1}{e^{kt}}}_{\to 0}\right] = \frac{1}{k^2}$

$p(x) = \frac{f(x)}{1/k^2} = \boxed{k^2 x e^{-kx}}$

b) $\mu = \int_0^\infty x\,p(x)\,dx = \int_0^\infty k^2 x^2 e^{-kx}\,dx = \lim_{t\to\infty} \frac{-1}{k}(k^2x^2+2kx+2)e^{-kx}\Big|_0^t$

$= \lim_{t\to\infty}\left[\frac{1}{k}(0+2)e^0 - \frac{1}{k}\underbrace{\frac{k^2t^2+2kt+2}{e^{kt}}}_{\to 0}\right] = \boxed{\frac{2}{k}} \to k = \frac{2}{\mu}$

$\boxed{p(x) = \frac{4}{\mu^2}\,x e^{-2x/\mu}} \xrightarrow{\mu=1} p(x) = 4xe^{-2x}$

c) $0 = p'(x) = 4(1e^{-2x} + x e^{-2x}(-2)) = 4e^{-2x}(1-2x) \to x = \boxed{\tfrac{1}{2} \equiv x_{max}}$

d) $P(0 \le x \le x_{max}) = \int_0^{1/2} 4xe^{-2x}\,dx = -(2x+1)e^{-2x}\Big|_0^{1/2} = (0+1)e^0 - (1+1)e^{-1}$

$= \boxed{1-2e^{-1} \approx 0.2642}$

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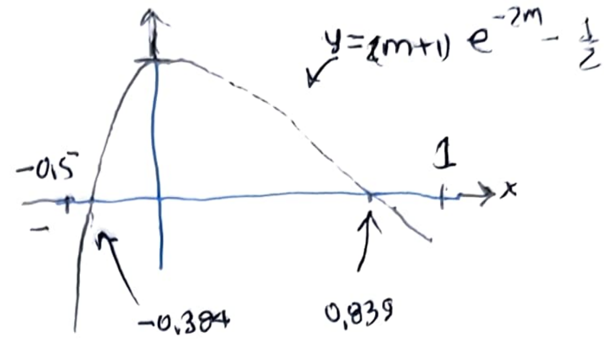
e) $\int_0^m 4xe^{-2x}\,dx = \frac{1}{2}$

$= -(2x+1)e^{-2x}\Big|_0^m = \underbrace{1e^0}_{1} - (2m+1)e^{-2m} = \frac{1}{2}$

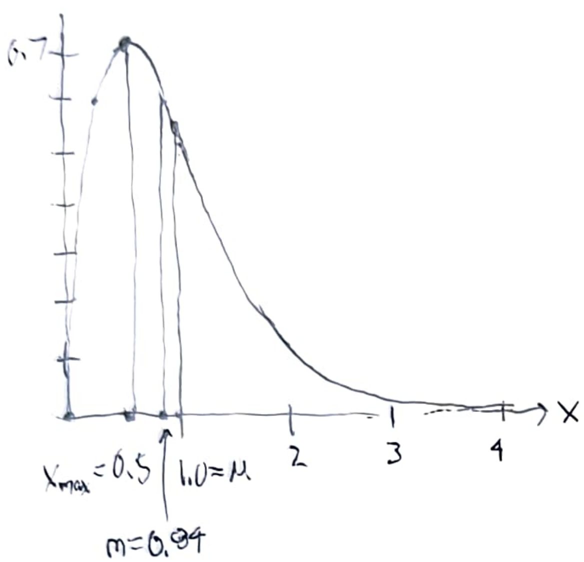
$\frac{1}{2} = (2m+1)e^{-2m} \xrightarrow[m>0]{} \boxed{m = 0.8392}$

Maple can solve this exactly!



$x_{max} = 0.5$, $m \approx 0.84$, $\mu = 1.0$

f)



always plot function whose zeros are sought numerically.
Solve, Numerical (without a starting pt) will give the negative root!
Use Solve, Numerical, from Point!

g) $\sigma^2 = \int_0^\infty (x-1)^2 P(x)\,dx = \int_0^\infty (x-1)^2 \cdot 4xe^{-2x}\,dx$

$= \lim_{t\to\infty} -\frac{1}{2}(4x^3-2x^2+2x+1)e^{-2x}\Big|_0^t = \frac{1}{2}(0+1) - 0 = \frac{1}{2}$

$\sigma = \frac{1}{\sqrt{2}} \approx 0.7071$

$1-\sigma \approx 0.2929$

$1+\sigma \approx 1.7071$

See Maple worksheet