MAT1505-05 22F Quiz 5 Print Name (Last, First) ,

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, IDENTIFYING expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation when appropriate). Indicate where technology is used and what type (Maple, GC). INDICATE where technology is used and what type (Maple, GC). Only use technology to CHECK hand calculations, not substitute for them.

The total energy in a cavity of volume V at temperature T of a Planck black body radiation gas (like the cosmic microwave background!) is given by

$$E_{thermal} = \frac{V\hbar}{\pi^2 c^3} \int_0^{\infty} \frac{\omega^3}{e^{\frac{\hbar\omega}{kT}} - 1}\, d\omega$$

where c is the speed of light, $\hbar$ is Planck's constant "h bar", T is the temperature of the radiation, k is Boltzman's constant, and ω is the frequency of the radiation. This is an integral over all frequencies of the energy per unit frequency using the Bose-Einstein distribution function.

a) Show that the obvious u-substitution $u = \frac{\hbar\omega}{kT} = \frac{\omega}{\omega_p}$, where $E_p = \hbar\omega_p = kT$ is the energy of one photon,

converts this formula to

$$E_{thermal} = K \int_0^{\infty} \frac{u^3}{e^u - 1}\, du .$$ What is the value of the constant K?

b) Use technology to evaluate the dimensionless definite integral factor. What is its value, exactly and numerically to 2 decimal places?

c) Combine the two factors and simplify the final formula for the energy (exactly). [The power law temperature dependence is a key feature of our understanding of the early universe.]

d) Plot the dimensionless integrand $f(u) = \frac{u^3}{e^u - 1}$ for $u = -1..1$, showing that this function has a "removable singularity" at the origin (make a rough sketch of what you see, labeling the axes) and becomes continuous if we define $f(0) = \lim_{u \to 0} \frac{u^3}{e^u - 1} = 0$, so its integral is well behaved at the left endpoint. Note that l'Hopital's rule leads to $\lim_{u \to 0} \frac{u^3}{e^u - 1} = \lim_{x \to 0} \frac{3u^2}{e^u} = 0$ so division by zero does not occur in the limit of the integrand.)

e) For very large u, the integrand looks like $u^3 e^{-u}$ so the original integral should have the same convergence properties there. Use Maple to evaluate $\int_1^{\infty} u^3 e^{-u}\, du$ to show that indeed this limiting integrant integral is finite. [The limit of the antiderivative at infinity is zero by 3 successive applications of l'Hopital's rule. State Maple's antiderivative for this simpler integral, which can also be written as a $\frac{\infty}{\infty}$ limit].

▶ solution

whystart here with so little space?

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a) $$E = \frac{V\hbar}{\pi^2 c^3}\int_0^\infty \frac{\omega^3}{e^{\frac{\hbar\omega}{kT}}-1}\,d\omega = \frac{V\hbar}{\pi^2 c^3}\int_0^\infty \frac{\left(\frac{kT}{\hbar}u\right)^3\left(\frac{kT}{\hbar}du\right)}{e^u-1}$$

$$u = \frac{\hbar\omega}{kT} \rightarrow \omega = \frac{kT}{\hbar}u$$

$$du = \frac{\hbar}{kT}\,d\omega$$

$\omega = 0 : u = 0$

$\omega = \infty : u = \infty$

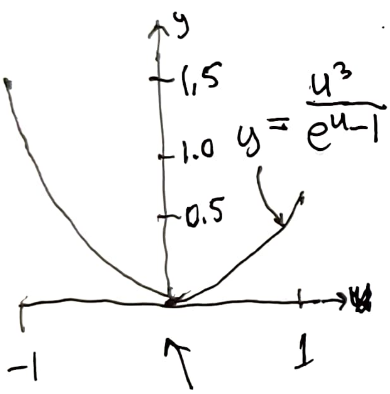
$$= \underbrace{\frac{V\hbar}{\pi^2 c^3}\left(\frac{kT}{\hbar}\right)^4}\ \underbrace{\int_0^\infty \frac{u^3\,du}{e^u-1}}$$

$$\boxed{K = \frac{V(kT)^4}{\pi^2(\hbar c)^3} = \frac{k^4VT^4}{\pi^2\hbar^3c^3}}$$

b) $\underset{\text{Maple}}{=} \boxed{\frac{\pi^4}{15}} \approx 6.49$

c) $$E = \frac{k^4VT^4}{\pi^2\hbar^3c^3}\left(\frac{\pi^4}{15}\right) = \boxed{\frac{\pi^2}{15}\,\frac{k^4VT^4}{\hbar^3c^3}}$$

d)



Clearly $\lim_{\omega\to 0} y = 0$

e) $$\int_1^\infty u^3 e^{-u}\,du \underset{\text{Maple}}{=} \boxed{16e^{-1}}$$

$$\int u^3 e^{-u}\,du = \boxed{-(u^3+3u^2+6u+6)e^{-u}} + C$$

maple's antiderivative.