MAT 1505-05 22F Test 3 Answers

① a) $$\sum_{n=0}^{\infty} \underbrace{\frac{(x-1)^n}{2^{n+1}(n+1)(n+3)}}_{a_n} = f(x)$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\dfrac{|x-1|^{n+1}}{2^{(n+1)+1}(n+1+1)(n+1+3)}}{\dfrac{|x-1|^n}{2^{n+1}(n+1)(n+3)}}$$

$$= \frac{|x-1|}{2}\,\frac{(n+1)(n+3)}{(n+2)(n+4)}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \Big(\underbrace{\lim_{n\to\infty}\frac{(n+1)(n+3)}{(n+2)(n+4)}}_{=\lim_{n\to\infty}\frac{n^2}{n^2}=1}\Big)\frac{|x-1|}{2} = \underbrace{\frac{|x-1|}{2} < 1}_{\text{ratio test}}$$

$|x-1| < \boxed{2 = R}$ radius of convergence

b) endpoints: $x-1 = \pm 2$
$x = -1, 3$

$x=-1$: $\frac{x-1}{2} = -1$

$$f(-1) = \sum_{n=0}^{\infty}\left(\frac{x-1}{2}\right)^n \frac{1}{2(n+1)(n+3)}$$

$$= \sum_{n=0}^{\infty} (-1)^n \underbrace{\frac{1}{2(n+1)(n+3)}}$$

$\frac{1}{2n^2}$ for large n

convergent p-series so converges absolutely! $(p=2>1)$

alternating sign, $|a_n| \to 0$ so converges by alternating series test

$x=3$: $\frac{x-1}{2} = 1$

$$f(3) = \sum_{n=0}^{\infty} \underbrace{\frac{1}{2(n+1)(n+3)}}$$

large n: $\sim \frac{1}{2n^2}$ convergent p-series as above, so converges at both endpoints

interval of convergence $\boxed{-1 \le x \le 3 \text{ or } [-1,3]}$

MAT1505-05 22F Test 3 Answers (2)

1 c) $f(0) = \sum_{n=0}^{\infty} (-1)^n \underbrace{\frac{1}{2^{n+1}(n+1)(n+3)}}_{b_n}$

$|b_6| = 0.000124 > .0001 \rightarrow \boxed{n=6}$

$|b_7| = |-0.0000491| < .0001$ next term

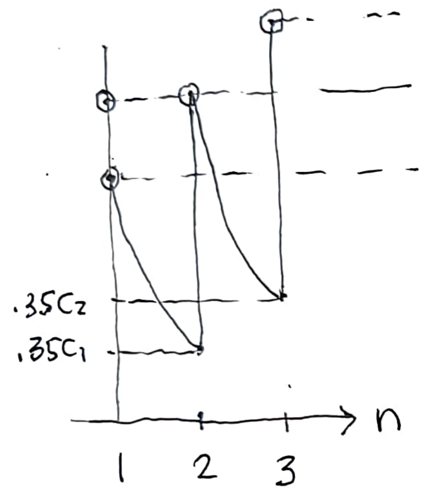
so $\sum_{n=0}^{6} b_n$ is the first partial sum within 0.0001 of the infinite series value.

d) Note:

$$\boxed{\begin{aligned} S &= 0.1418\,023 \\ S_6 &= 0.1419\,372 \\ \hline S - S_6 &= -0.0000\,349 \end{aligned}} \leftarrow |S - S_6| < 0.0001$$

(Maple evaluations)

2 a)



$c_3 = .35(.35(1.5) + 1.5) + 1.5 = \boxed{2.209}$

$c_2 = (.35)(1.5) + (1.5) = 2.025$

$c_1 = 1.5$

b) $c_3 = 1.5 + 1.5(0.35) + 1.5(0.35)^2$

clearly

$$\boxed{c_n = \sum_{k=1}^{n} \underbrace{1.5}_{a}\,(\underbrace{0.35}_{r})^{k-1} = \frac{a(1-r^n)}{1-r}}$$

geometric series

c) so $c_\infty = \frac{a}{1-r} = \frac{1.5}{1-0.35} = \frac{1.5}{0.65} = \frac{150}{65} = \frac{30}{13} \approx 2.308 \approx \boxed{2.31}$

$0.99\, c_\infty = 2.285$

d) $c_4 = 2.273$

$c_5 = 2.296$

(Maple)

$\boxed{n=5}$ is the first injection reaching within 1% of the asymptotic value

e) Subtract the dose:

$$c_{min} = c_\infty - a = \begin{array}{r} 2.308 \\ -1.500 \\ \hline 0.808 \end{array} \approx \boxed{0.81}$$

is the asymptotic minimum concentration.

MAT1505-05 22F Test 3 Answers (3)

③ a) $V = 2\pi k_e \sigma \left(\underbrace{\sqrt{d^2+R^2}}_{\substack{\sqrt{d^2\left(1+\frac{R^2}{d^2}\right)} \\ d\sqrt{1+\left(\frac{R}{d}\right)^2}}} - d\right) = 2\pi k_e \sigma d\left(\left(1+\left(\frac{R}{d}\right)^2\right)^{1/2} - 1\right)$

$(1+x)^k$

$x = \left(\frac{R}{d}\right)^2, \ k = \frac{1}{2}$

binomial series

$= 2\pi k_e \sigma d\left(\cancel{1} + \frac{1}{2}\left(\frac{R}{d}\right)^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1\cdot 2}\left(\frac{R}{d}\right)^4 + \ldots \cancel{-1}\right)$ 0

$= 2\pi k_e \sigma d\left(\underbrace{\frac{R^2}{2d^2} - \frac{1}{8}\frac{R^4}{d^4} + \ldots}_{\underbrace{\frac{R^2}{2d^2}}\left(1 - \frac{R^2}{4d^2} + \ldots\right)}\right)$ ← alternating series;

$\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\ldots$

$(+)\,(-1)\quad(-1)(-1)$

changes sign each term

$= \frac{\pi k_e \sigma R^2}{d}\left(1 - \underbrace{\frac{R^2}{4d^2}} + \ldots\right) \longrightarrow \boxed{\frac{\pi k_e \sigma R^2}{d}}$ at lowest order for $R \ll d$

b) estimate for fractional error; set equal to 0.01 :

$\frac{R^2}{4d^2} = \frac{1}{100} \rightarrow \frac{R^2}{d^2} = \frac{1}{25} \rightarrow \frac{R}{d} = \frac{1}{5}$ or $d = 5R$ ✓

c) $V_{approx} = \frac{\pi k_e \sigma R^2}{d}$

$100 * \left(\frac{V_{approx} - V}{V}\right)\Big|_{d=5R} \overset{\text{Maple}}{=} 0.9902\%$ (percentage error)

$< 1\%$! confirms estimate!