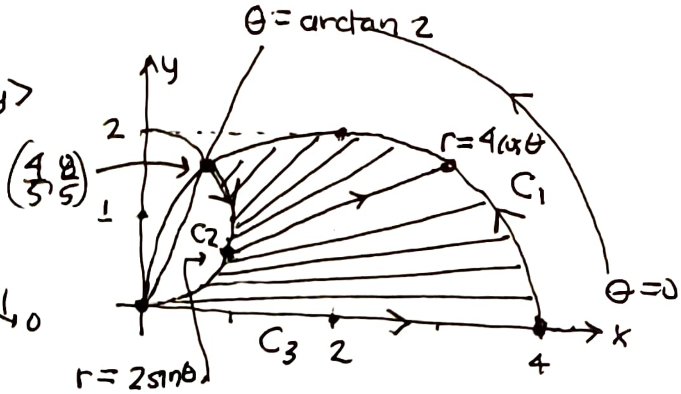
MAT 2500-01/03 21S Final Exam Answers (1)

① $\int_C -x^2dx + xy\,dy = \int_C \underbrace{\langle -x^2, xy\rangle}_{\vec{F}(x,y)} \cdot \langle dx, dy\rangle$



a) C_1: $(x-2)^2+y^2=4$ $\qquad$ C_2: $x^2+(y-1)^2=1$

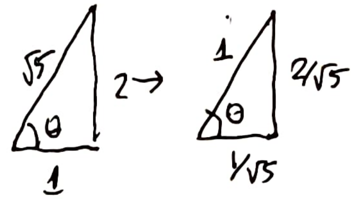
$\underbrace{x^2+y^2}_{r^2}-\underbrace{4x}_{r\cos\theta}+\cancel{4}=\cancel{4}_0$ $\qquad$ $\underbrace{x^2+y^2}_{r^2}-\underbrace{2y}_{r\sin\theta}+\cancel{1}=\cancel{1}_0$

$r^2=4r\cos\theta$ $\qquad$ $r^2=2r\sin\theta$

$r=4\cos\theta \longrightarrow$ intersection $\leftarrow r=2\sin\theta$

$4\cos\theta = 2\sin\theta$

$2=\tan\theta$, $\boxed{\theta=\arctan 2}$:



$\theta = \arcsin 2/\sqrt{5}$
$= \arccos 1/\sqrt{5}$

$r=4\cos\theta$
$=4\left(\frac{1}{\sqrt5}\right)=\boxed{\frac{4}{\sqrt5}}$

b) C_1: $r=4\cos\theta$ while $\theta=0..\arctan 2$

C_2: $r=2\sin\theta$ while $\theta=\arctan 2..0$

C_3: $x=t, y=0$ while $t=0..4$

$x=r\cos\theta=\left(\frac{4}{\sqrt5}\right)\left(\frac{1}{\sqrt5}\right)=\boxed{\frac45}$

$y=r\sin\theta=\left(\frac{4}{\sqrt5}\right)\left(\frac{2}{\sqrt5}\right)=\boxed{\frac85}$

b) $\vec{r}=\langle x,y\rangle=\langle r\cos\theta, r\sin\theta\rangle$ $\qquad$ $\vec{F}(x,y)=\langle -x^2, xy\rangle$

C_1: $\vec{r}=\langle (4\cos\theta)\cos\theta, (4\cos\theta)\sin\theta\rangle = 4\langle \cos^2\theta, \sin\theta\cos\theta\rangle$

$\vec{r}\,'=4\langle -2\cos\theta\sin\theta, \cos^2\theta-\sin^2\theta\rangle$, $\vec{F}(\vec{r}(\theta))=\langle -16(\cos^2\theta)^2, 16(\cos^2\theta)(\sin\theta\cos\theta)\rangle$

$\vec{r}\,'\cdot\vec{F}(\vec{r}(\theta)) = 64(2\cos^5\theta\sin\theta + (\cos^2\theta-\sin^2\theta)(\cos^3\theta\sin\theta))$

$= 64(3c^5s - c^3s^4)$

C_2: $\vec{r}=\langle (2\sin\theta)\cos\theta, (2\sin\theta)\sin\theta\rangle = 2\langle \sin\theta\cos\theta, \sin^2\theta\rangle$

$\vec{r}\,'=2\langle \cos^2\theta-\sin^2\theta, 2\sin\theta\cos\theta\rangle$, $\vec{F}(\vec{r}(\theta))=\langle -4(\sin\theta\cos\theta)^2, 4\sin^2\theta(\sin\theta\cos\theta)\rangle$

$\vec{r}\,'\cdot\vec{F}(\vec{r}(\theta)) = 8(-(\cos^2\theta-\sin^2\theta)\sin^2\theta\cos^2\theta + 2\sin^4\theta\cos^2\theta) = 8(-s^2c^4+3s^4c^2)$

C_3: $\vec{r}=\langle t,0\rangle$, $\vec{r}\,'=\langle 1,0\rangle$, $\vec{F}(\vec{r}(t))=\langle -t^2, 0\rangle$

$\vec{r}\,'\cdot\vec{F}(\vec{r}(t)) = -t^2$

$\int_{C_1}\vec{F}\cdot d\vec{r} = \int_0^{\arctan 2} 64(3\cos^5\theta\sin\theta - \cos^3\theta\sin^4\theta)\,d\theta \underset{\text{Maple}}{=} \frac{10112}{375} \approx 26.9653$

$\int_{C_2}\vec{F}\cdot d\vec{r} = \int_{\arctan 2}^0 8(-\sin^2\theta\cos^4\theta + 3\sin^4\theta\cos^2\theta)\,d\theta \underset{\text{Maple}}{=} \frac{38}{375} - \arctan 2 \approx -1.00581$

$\int_{C_3}\vec{F}\cdot d\vec{r} = \int_0^4 -t^2\,dt = -\frac{t^3}{3}\Big|_0^4 = -\frac{64}{3} \approx -21.333$

$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1}\vec{F}\cdot d\vec{r} + \int_{C_2}\vec{F}\cdot d\vec{r} + \int_{C_3}\vec{F}\cdot d\vec{r} \underset{\text{Maple}}{=} \boxed{\frac{86}{15} - \arctan 2 \approx 4.62618}$

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① c) $r = 2\sin\theta \,..\, 4\cos\theta$ while $\theta = 0 \,..\, \arctan 2$ (see diagram on page1)

$\vec{F} = \langle -x^2, xy \rangle$

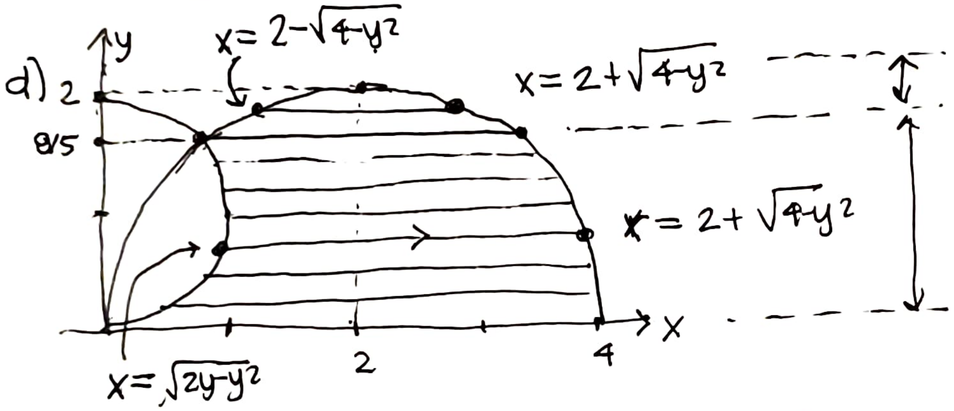
$\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} = \dfrac{\partial}{\partial x}(xy) - \dfrac{\partial}{\partial y}(-x^2) = y - 0 = y = r\sin\theta$

$$\iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dA = \int_0^{\arctan 2} \underbrace{\int_{2\sin\theta}^{4\cos\theta} \underbrace{(r\sin\theta)\, r}_{r^2\sin\theta}\, dr}_{\frac{r^3}{3}\sin\theta \Big|_{r=2\sin\theta}^{r=4\cos\theta} = \frac{64}{3}\cos^3\theta\sin\theta - \frac{8}{3}\sin^3\theta\sin\theta} d\theta$$

$$= \int_0^{\arctan 2} \frac{64}{3}\cos^3\theta\sin\theta - \frac{8}{3}\sin^4\theta\, d\theta .$$

$$\underset{\text{Maple}}{=} \boxed{\frac{86}{15} - \arctan 2}$$ agrees with line integral

d)



$x = 2-\sqrt{4-y^2} \,..\, 2+\sqrt{4-y^2}$ while $y = 8/5 \,..\, 2$

$x = \sqrt{2y-y^2} \,..\, 2+\sqrt{4-y^2}$ while $y = 0 \,..\, 8/5$

C_1: $(x-2)^2 + y^2 = 4$

$(x-2)^2 = 4 - y^2$

$x - 2 = \pm\sqrt{4-y^2}$

$x = 2 \pm \sqrt{4-y^2} \to 2 + \sqrt{4-y^2}$ (right)

C_2: $x^2 + (y-1)^2 = 1$

$x^2 + y^2 - 2y + \cancel{1 - 1}\,0$

$x^2 = 2y - y^2$

$x = \pm\sqrt{2y-y^2} \to \sqrt{2y-y^2}$ (right)

$$\iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dA = \int_0^{8/5}\int_{\sqrt{2y-y^2}}^{2+\sqrt{4-y^2}} y\, dx\, dy + \int_{8/5}^{2}\int_{2-\sqrt{4-y^2}}^{2+\sqrt{4-y^2}} y\, dx\, dy$$

$$\underset{\text{Maple}}{=} \boxed{\frac{86}{15} - \underbrace{\arcsin\left(\frac{2}{\sqrt{5}}\right)}_{=\arctan 2} = \frac{86}{15} - \arctan 2}$$ agrees with polar integral

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① e) optional.

$\vec{F} = \frac{1}{2}\langle -y, x\rangle \qquad \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{\partial}{\partial x}\left(\frac{x}{2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{2}\right) = \frac{1}{2}(1+1) = 1$

so $\oint \vec{F}\cdot d\vec{r} = \iint_R 1\, dA = A$ (area)

C_1: $\vec{F}(\vec{r}(\theta)) = \frac{1}{2}\langle -4sc, 4c^2\rangle$, $\vec{r}\,' = \langle 2c^2s^2, 4sc\rangle$

$\vec{F}(\vec{r}(\theta))\cdot\vec{r}\,' = \langle -2sc, 2c^2\rangle\cdot\langle 2c^2s^2, 4sc\rangle = -4s^3c^3 + 8sc^3$

$\int_{C_1}\vec{F}\cdot d\vec{r} = \int_0^{\arctan 2} -4\sin^3\theta\cos^3\theta + 8\sin\theta\cos^3\theta\, d\theta = \frac{8}{5} + 4\arctan 2$

$\approx$

C_2: $\vec{F}(\vec{r}(\theta)) = \frac{1}{2}\langle -2s^2, 2sc\rangle$, $\vec{r}\,' = \langle 2sc, 2s^2\rangle$

$\vec{F}(\vec{r}(\theta))\cdot\vec{r}\,' = \langle -s^2, sc\rangle\cdot\langle 2sc, 2s^2\rangle = -2s^3c + 2s^3c$

$\int_{C_2}\vec{F}\cdot d\vec{r} = \int_{\arctan 2}^{0} -2\sin^3\theta\cos\theta + 2\sin^3\theta\cos\theta\, d\theta = \frac{2}{5} - \arctan 2$

C_3: $\vec{F}(\vec{r}(t)) = \frac{1}{2}\langle 0, t\rangle \qquad \vec{r}\,' = \langle 1, 0\rangle \qquad \vec{F}(\vec{r}(t))\cdot\vec{r}\,' = 0 + 0 = 0$

$\int_{C_3}\vec{F}\cdot d\vec{r} = \int_0^4 0\, dt = 0$

$\oint_C \vec{F}\cdot d\vec{r} = \int_{C_1}\vec{F}\cdot d\vec{r} + \int_{C_2}\vec{F}\cdot d\vec{r} + \int_{C_3}\vec{F}\cdot d\vec{r} = \left(\frac{8}{5} + 4\arctan 2\right) + \left(\frac{2}{5} - \arctan 2\right) + 0$

$= \boxed{2 + 3\arctan 2 \approx 5.321447}$

- in polar coords:

$A = \iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dA = \iint_R 1\, dA = \int_0^{\arctan 2}\int_{2\sin\theta}^{4\cos\theta} 1\cdot r\, dr\, d\theta$

$\underset{\text{Maple}}{=} \boxed{2 + 3\arctan 2}$ agrees with line integral

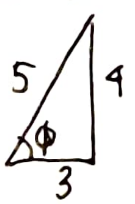
- in cartesian coords:

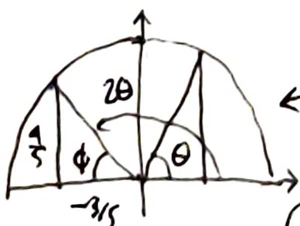
$A = \int_0^{8/5}\int_{\sqrt{2y-y^2}}^{2+\sqrt{4-y^2}} 1\, dx\, dy + \int_{8/5}^{2}\int_{2-\sqrt{4-y^2}}^{2+\sqrt{4-y^2}} 1\, dx\, dy \underset{\text{Maple}}{=} 2 + \overbrace{2\pi - 2\arcsin\frac{4}{5}}^{4\arctan 2 \text{ (see below)}}$

$\underbrace{\arcsin 2/\sqrt{5}}$

$= \arctan 2$

$\phi = \arcsin 4/5$:





$\pi - \phi = 2\theta$

double angle formulas:

$\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{2}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{5}}\right) = \frac{4}{5}$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = \left(\frac{1}{\sqrt{5}}\right)^2 - \left(\frac{2}{\sqrt{5}}\right)^2 = -\frac{3}{5}$

so $\pi - \arcsin 4/5 = 2\arctan 2$

$2\pi - 2\arcsin 4/5 = 4\arctan 2$

$= \boxed{2 + 3\arctan 2}$

agrees !!

② a) $\vec{r} = \langle \cos t, \sin t, \sin 2t \rangle$

$\vec{r}\,' = \langle -\sin t, \cos t, 2\cos 2t \rangle$

$\vec{F}(x,y,z) = \langle yz, zx, xy \rangle$

$\vec{F}(\vec{r}(t)) = \langle \sin t \sin 2t, \sin 2t \cos t, \sin t \cos t \rangle$

$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = (\sin t \sin 2t)(-\sin t) + (\sin 2t \cos t)(\cos t) + (\sin t \cos t)(2\cos 2t) = \sin 2t\,(\cos^2 t - \sin^2 t) + \cos 2t\,(2\sin t \cos t)$

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = \int_0^{2\pi} \sin 2t\,\underbrace{(\cos^2 t - \sin^2 t)}_{\cos 2t} + \cos 2t\,\underbrace{(2\sin t \cos t)}_{\sin 2t}\,dt = 0$$

Maple

$$\left(= \int_0^{2\pi} \underbrace{\sin 2t}_{u}\ \underbrace{\cos 2t\,dt}_{du/2} \to u^2/4 \quad = \sin^2 2t \Big|_0^{2\pi} = 0! \right)$$

we can easily do this by hand!

b) $\vec{r}(0) = \langle \cos 0, \sin 0, \sin 0 \rangle = \langle 1, 0, 0 \rangle$

$\vec{r}(\frac{\pi}{3}) = \langle \cos\frac{\pi}{3}, \sin\frac{\pi}{3}, \sin\frac{2\pi}{3} \rangle = \langle \frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2} \rangle$ $\quad \vec{r}(\frac{\pi}{3}) - \vec{r}(0) = \langle -\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2} \rangle$

$\vec{r} = \langle x,y,z \rangle = \vec{r}(0) + t(\vec{r}(\pi/3) - \vec{r}(0)) = \langle 1,0,0 \rangle + \frac{t}{2}\langle -1, \sqrt{3}, \sqrt{3} \rangle$

$= \langle 1 - t/2, t\sqrt{3}/2, t\sqrt{3}/2 \rangle \qquad \vec{r}\,'(t) = \langle -\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2} \rangle$

$\vec{F}(\vec{r}(t)) = \langle \underbrace{\left(t\frac{\sqrt{3}}{2}\right)^2}_{yz}, \underbrace{\left(1-\frac{t}{2}\right)}_{x}\underbrace{\left(t\frac{\sqrt{3}}{2}\right)}_{z}, \underbrace{\left(1-\frac{t}{2}\right)}_{x}\underbrace{\left(t\frac{\sqrt{3}}{2}\right)}_{y} \rangle$

$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = \left(t^2\frac{3}{4}\right)\left(-\frac{1}{2}\right) + \left(1-\frac{t}{2}\right)\left(\frac{\sqrt{3}t}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(1-\frac{t}{2}\right)\left(t\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$

$= -\frac{3}{8}t^2 + \frac{3}{8}(2-t)t + \frac{3}{8}(2-t)t$

$= \frac{3}{8}(-t^2 + 2t - t^2 + 2t - t^2) = \frac{3}{8}(-3t^2 + 4t)$

$= -\frac{9}{8}t^2 + \frac{3}{2}t$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^1 \frac{3}{2}t - \frac{9}{8}t^2\,dt = \frac{3}{4}t^2 - \frac{3}{8}t^3 \Big|_0^1 = \frac{3}{4} - \frac{3}{8} = \boxed{\frac{3}{8}}$$

c) $\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(zx) + \frac{\partial}{\partial z}(xy) = 0 + 0 + 0 = \boxed{0}$

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3 a) $\vec{F}(x,y,z) = \langle 2x\ln y,\ \frac{x^2}{y}+z^2,\ 2yz\rangle$

C: $\vec{r}(t) = \langle t^2, t, t\rangle$, $t = 1..e$

$\vec{r}\,'(t) = \langle 2t, 1, 1\rangle$

$\vec{F}(\vec{r}(t)) = \langle 2t^2\ln t,\ (t^2)^2/t + t^2,\ 2t(t)\rangle = \langle 2t^2\ln t,\ t^3+t^2,\ 2t^2\rangle$

$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = (2t^2\ln t)(2t) + (t^3+t^2)(1) + (2t^2)(1)$

$= 4t^3\ln t + t^3 + 3t^2$

$$\int_C \vec{F}\cdot d\vec{r} = \int_1^e 4t^3\ln t + t^3 + 3t^2\,dt \underset{\text{Maple}}{=} \boxed{e^4+e^3-1} \approx 73.684$$

b) $\operatorname{curl}\vec{F} = \langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z},\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x},\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\rangle$

$= \langle \frac{\partial}{\partial y}(2yz) - \frac{\partial}{\partial z}(\frac{x^2}{y}+z^2),\ \frac{\partial}{\partial z}(2x\ln y) - \frac{\partial}{\partial x}(2yz),\ \frac{\partial}{\partial x}(\frac{x^2}{y}+z^2) - \frac{\partial}{\partial y}(2x\ln y)\rangle$

$= \langle 2z - (0+2z),\ 0 - 0,\ \frac{2x}{y} + 0 - \frac{2x}{y}\rangle = \langle 0,0,0\rangle = \vec{0}$

c) $\vec{F} = \vec{\nabla} f$: $\langle F_1, F_2, F_3\rangle = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\rangle$

$\int \left[\frac{\partial f}{\partial x} = 2x\ln y\right] dx \rightarrow$ $f = \int 2x\ln y\,dx = x^2\ln y + C(y,z)$

$\frac{\partial f}{\partial y} = \frac{x^2}{y}+z^2$ $\qquad \frac{\partial f}{\partial y} = \frac{x^2}{y} + \frac{\partial C}{\partial y}(y,z) = \frac{x^2}{y}+z^2$

$\frac{\partial f}{\partial z} = 2yz$

$\int \left[\frac{\partial C}{\partial y}(y,z) = z^2\right] dy$

$\left[C(y,z) = \int z^2\,dy = z^2y + C(z)\right]$

$f = x^2\ln y + yz^2 + C(z)$

$\frac{\partial f}{\partial z} = 2yz + C'(z) = 2yz$

$C'(z) = 0$

$C(z) = k$

$\boxed{f = x^2\ln y + yz^2 + k}$

$\vec{r}(t) = \langle t^2, t, t\rangle$

d) $\vec{r}(1) = \langle 1,1,1\rangle$

$\vec{r}(e) = \langle e^2, e, e\rangle$

$f(e^2,e,e) - f(1,1,1) = (e^2)^2\overbrace{\ln e}^{1} + e\,e^2 + k - [1^2\underbrace{\ln 1}_{0} + 1\cdot 1^2 + k] = e^4(1) + e^3 - 1 = \boxed{e^4+e^3-1}$ ✓