MAT2500-01/03 21S Test 3 Print Name (Last, First) ____________________________

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation IF appropriate). Indicate where technology is used and what type (Maple, GC). Only use technology to CHECK hand calculations, not substitute for them unless otherwise specified. Explain in as many words as possible everything you are doing! For each hand integration step, state the antiderivative formula used before substituting limits into it: $\int_a^b f(x)\,dx = F(x)\big|_{x=a}^{x=b} = F(b) - F(a)$.

pledge

When you have completed the exam, please read and sign the dr bob integrity pledge and **and scan** these test sheets as a cover first and second page in the PDF scan of your lined paper hand work, **all done on separate sheets.**

"During this examination, all work has been my own. I give my word that I have not resorted to any ethically questionable means of improving my grade or anyone else's on this examination and that I have not discussed this exam with anyone other than my instructor, nor will I until after the exam period is terminated for all participants."

Signature: Date:

1. Consider the integral $A = \iint_R 1\, dA$ over the region of the plane with bounding curves $x = 1, y = 1, x = e^y$.

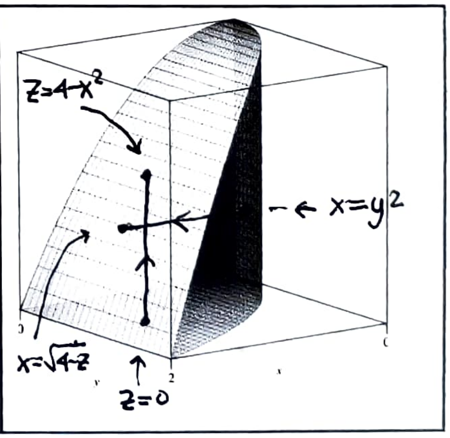
a) Make a diagram of this region with equally spaced linear cross-sections illustrating the x first iteration of the integral, with a typical bullet point terminated linear cross-section (arrowhead in the middle) annotated by the starting and stopping value equations of the inner integration variable, while the outer variable of integration linits should be clearly marked. Set up the double integral with this iteration and evaluate it step by step.
b) Repeat in a new copy of your diagram for the opposite order of integration.
c) For which of these integrals did you not need help for the anti-derivative?
d) If you use Maple to evaluate exactly the integral of $\cos\left(\frac{2y}{\pi}\right)$ over this region, which order yields a much shorter expression for the result? Do they both give the same numerical value? What is that value to 4 decimal places?

$$\cos\left(\frac{2y}{\pi}\right) \qquad (1)$$

2. Consider the solid region R in the first octant corresponding to the region of integration in the triple integral

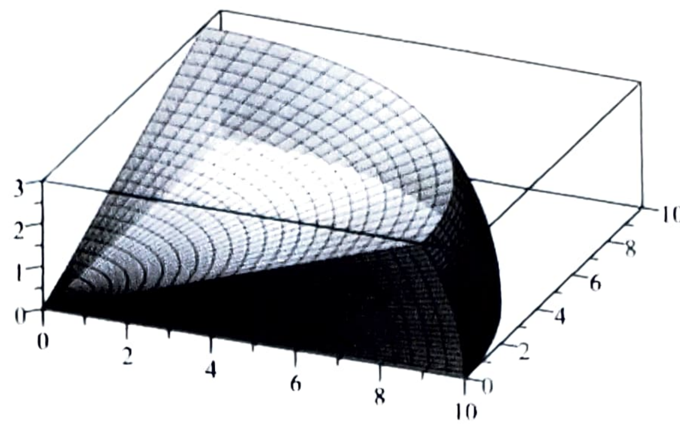
$$\int_0^2 \int_0^{\sqrt{x}} \int_0^{4-x^2} f(x, y, z)\, dz\, dy\, dx .$$

a) Evaluate this integral by hand step by step with $f(x, y, z) = 1$ to get the volume of the region.
b) Make a labeled plane diagram of the projection of R onto the x-y plane (with a labeled cross-section for the inner integration of the outer double integral) and indicate in the 3-d diagram a typical labeled bullet point terminated linear cross-section indicating the inner most integral (labeled by the starting and stopping value equations for the variable of integration)

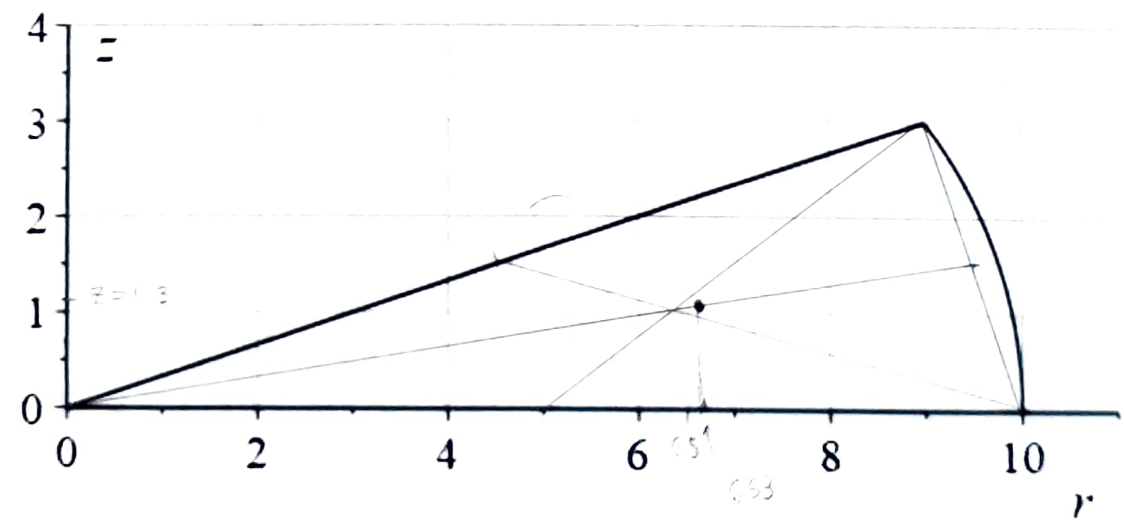


c) Rewrite the integral in the order $\iiint \ldots\, dx\, dz\, dy$ by first making a labeled plane diagram of the projection of R onto the y-z plane (with a labeled bullet point terminated cross-section for the inner integration of the outer double integral) and indicate in the 3-d diagram a typical labeled bullet point terminated linear cross-section indicating the inner most integral (labeled by the starting and stopping value equations for the variable of integration). to get the volume of this region. Your results should agree.

3. A pinched torus can be obtained by rotating a circle in the r-z half plane centered at $(r, z) = (5, 0)$ of radius 5: $(r-5)^2 + z^2 = 25$. The wedge of this torus between the upper cone: $9\,z^2 = x^2 + y^2,\ z \geq 0$ and the x-y plane is the solid we are studying, shown to the right.
a) Find the (r, z) coordinates of the intersection of the cone and torus and make a fully annotated diagram of these three surfaces in the r-z half plane cross-section of this solid.



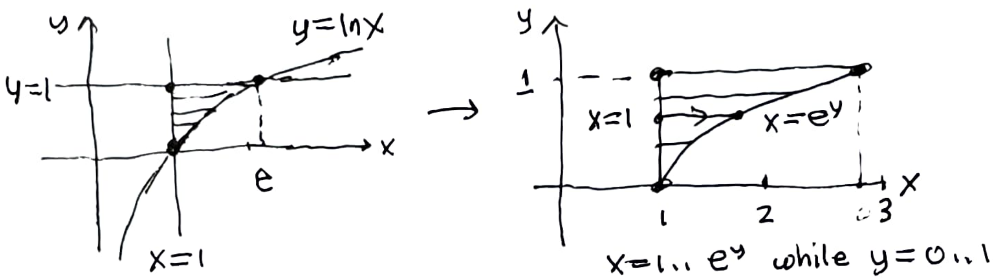
b) Annotate your diagram to illustrate a radial first triple integral of the volume of this region in cylindrical coordinates.
c) Write down the corresponding triple integral for the volume and evaluate it with Maple exactly and numerically to 2 decimal places.
d) Make a new diagram of this region in the r-z half plane using spherical coordinates, with annotations for the radial first triple integral in those coordinates.
e) Write down the corresponding triple integral for the volume and evaluate it with Maple exactly and numerically to 2 decimal places.
[f) **Optional Challenge (tedious arithmetic).** Evaluate the volume of the torus step by step in spherical coordinates, using Maple to state the necessary antiderivatives.]
g) Evaluate and use Maple to evaluate the 3 moments in spherical coordinates [call them (Vyz, Vxz, Vxy)] needed to evaluate the centroid $(\overline{x}, \overline{y}, \overline{z})$ of this solid to 2 decimal place accuracy, and state the numerical values of the centroid (Cartesian) coordinates.
h) By rotational symmetry the centroid will lie in the central value of the interval in θ. Evaluate the cylindrical coordinates of this point in the r-z half plane for that value of θ and plot it in the fully annotated diagram below. This should actually be the centroid of the 2-d cross section. Does it look sort of in the right place? Draw in the secant line between the endpoints of the torus arc in the figure below and then in the triangle it forms, use a ruler to draw in the three perpendicular bisectors to find graphically their intersection point. Does the centroid move from this triangle centroid in the right direction? Explain.



Optional 5pts. Find the centroid of the triangle by solving for the intersection point of the bisector lines and create a Maple plot of the above cross-section with the three bisectors shown together with the torus wedge centroid, and see where the latter lies with respect to the bisector lines.

MAT 2500-01/03 21S Test 3 Answers (1)

① a) $x=1,\ y=1,\ x=e^y \Leftrightarrow y=\ln x$

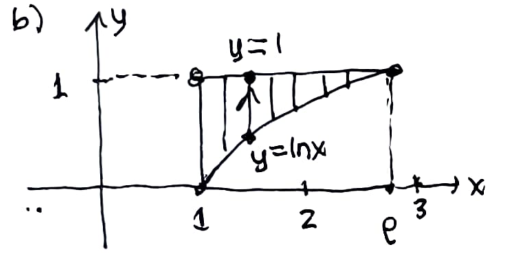


$x=1..e^y$ while $y=0..1$

$$A=\int_0^1 \underbrace{\int_1^{e^y} 1\,dx}_{x\big|_{x=1}^{x=e^y}=e^y-1}\,dy = \int_0^1 e^y-1\,dy = e^y-y\Big|_0^1 = e^1-1-(1-0)=e-2$$

$$\approx 0.718$$

b)



$y=\ln x..1$ while $x=1..e$

$$A=\int_1^e \underbrace{\int_{\ln x}^1 1\,dy}_{y\big|_{y=\ln x}^{y=1}=1-\ln x}\,dx = \int_1^e 1-\ln x\,dx$$

$$= x-(x\ln x - x)\Big|_1^e \quad \text{Maple}$$

$$= 2x - x\ln x\Big|_1^e = 2e-2-e\underbrace{\ln e}_{1}+1\cancel{\ln 1}^{0}$$

$$= e-2 \ \checkmark$$

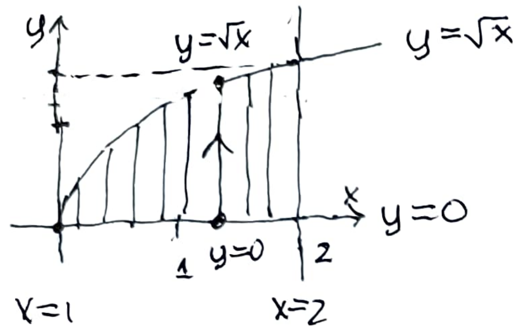
c) → This order needed integration by parts, blech!

d) The x first integral is about half as long as the y first integral expression. If you numerically evaluate them to N=10, then N=12 digits, you see both are too high in the last 2 digits at N=10, but the first one is closer to the correct 10 digit accurate result.
When Maple has to combine many more operations in evaluating each term/factor to a given number of digits, the final digits lose accuracy!

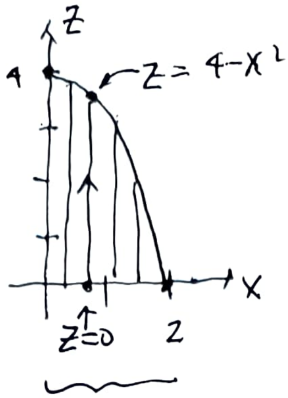
To 4 digits: 0.6421

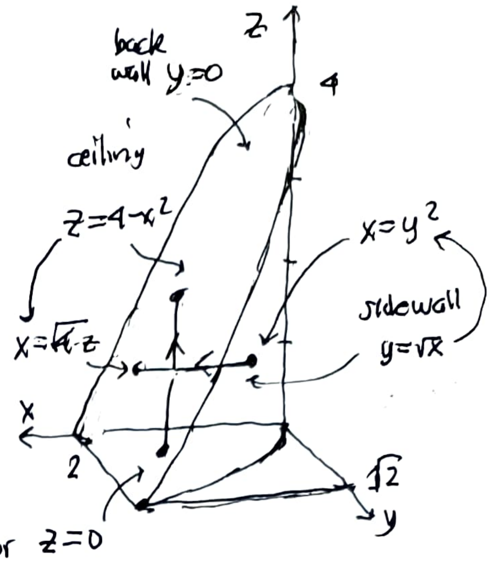
MAT2500-OV03 21S Test 3 Answers (2)

(2) $\displaystyle\int_{x=0}^{x=2}\int_{y=0}^{y=\sqrt{x}}\int_{z=0}^{z=4-x^2} f \, dz\,dy\,dx$



$y=0..\sqrt{x}$ while $x=0..2$





3d diagrams

c) ceiling, sidewall intersect:

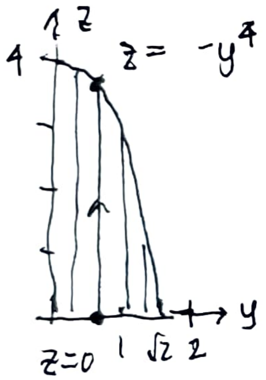
$z = 4-x^2 = 4-y^4$ while $y=0..\sqrt{2}$

$x=y^2$

$$V = \int_0^{\sqrt{2}}\int_0^{4-y^4}\int_{y^2}^{\sqrt{4-z}} 1 \, dx\,dz\,dy$$

$$= \frac{64\sqrt{2}}{21}$$

agrees



$z=0..4-y^4$ while $y=0..\sqrt{2}$

start here:

a) $\displaystyle\int_0^2\int_0^{\sqrt{x}}\underbrace{\int_0^{4-x^2} 1 \, dz}_{z\big|_{z=0}^{z=4-x^2} = 4-x^2} dy\,dx = \int_0^2 \underbrace{\int_0^{\sqrt{x}} 4-x^2 \, dy}_{4y-x^2y\big|_{y=0}^{y=\sqrt{x}} = 4x^{1/2}-x^{5/2}} dx$

$$= \int_0^2 4x^{1/2}-x^{5/2} \qquad = 4\cdot\frac{2}{3}x^{3/2} - \frac{2}{7}x^{7/2}\Big|_{x=0}^{x=2}$$

$$= \frac{1}{3}2^{3+3/2} - \frac{1}{7}2^{1+7/2}$$

$$= \frac{1}{3}2^{9/2} - \frac{1}{7}2^{9/2} = \frac{7-3}{21}2^4\sqrt{2} = \frac{64}{21}\sqrt{2}$$ Maple agrees!

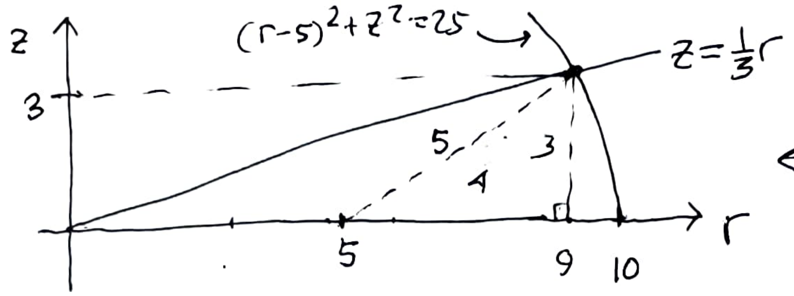
$$\approx 4.30998$$

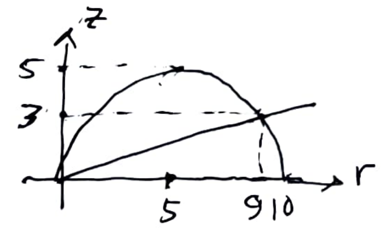
MAT2500-01/03 21S Test 3 Answers (3)

3 a) $(r-5)^2+z^2=25 \to r^2-10r+25+z^2=25 \to r^2+z^2-10r=0$

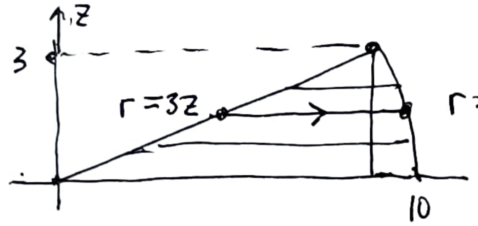
$9z^2=x^2+y^2=r^2 \to z=\frac{1}{3}r \quad (z\ge 0)$ — intersect

$r^2+\frac{1}{9}r^2-10r=0 \to \frac{10}{9}r^2=10r \to \boxed{r=9 \to z=3}$





b)



$r^2-10r+z^2=0$

$r=\frac{10\pm\sqrt{100-4z^2}}{2}$

$=5\pm\sqrt{25-z^2}$

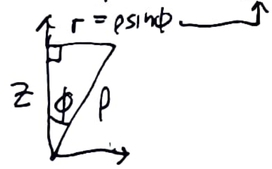
+ root for right semicircle.

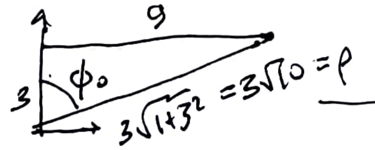
$r=3z\,..\,5+\sqrt{25-z^2}$ while $z=0\,..\,3$

c) $V=\int_0^{\pi/2}\int_0^3\int_{3z}^{5+\sqrt{25-z^2}} \underbrace{r\,dr\,dz\,d\theta}_{dV}$

$=\frac{\pi}{2}\left(60+\frac{125}{2}\arcsin\frac{3}{5}\right)\approx 157.42$

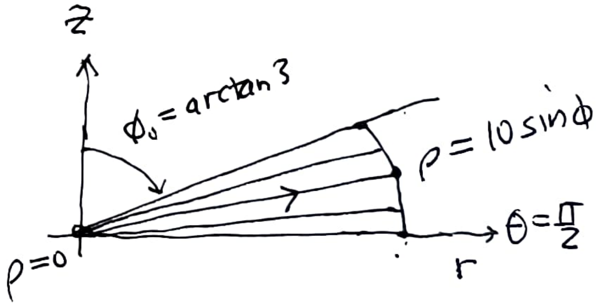
d) $\underbrace{r^2+z^2}_{\rho^2}=10\underbrace{r}_{\rho\sin\phi} \to \rho=10\sin\phi$





$\tan\phi_0=3,\ \phi_0=\arctan 3$

$\sin\phi_0=\frac{3}{\sqrt{10}}$, $\cos\phi_0=\frac{1}{\sqrt{10}}$ $\Big\}$ $\phi_0\approx 71.6°$



e) f) $V=\int_0^{\frac{\pi}{2}}\int_{\phi_0}^{\frac{\pi}{2}}\int_0^{10\sin\phi} 1\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=\frac{\pi}{2}\int_{\phi_0}^{\frac{\pi}{2}}\int_0^{10\sin\phi}\rho^2\sin\phi\,d\rho\,d\phi$

$=\frac{\pi}{2}\int_0^{\pi/2}\frac{\rho^3}{3}\sin\phi\,\Big|_{\rho=0}^{\rho=10\sin\phi}d\phi=\frac{\pi}{2}\int_{\phi_0}^{\frac{\pi}{2}}\frac{10^3}{3}\sin^4\phi\,d\phi$

Maple $=\frac{\pi}{2}\cdot\frac{10^3}{3}\left[\underbrace{-\left(\frac{1}{4}\sin^3\phi+\frac{3}{2}\sin\phi\right)\cos(\phi)}_{-\frac{1}{4}(\sin^2\phi+6)\sin\phi\cos\phi}+\frac{3}{8}\phi\right]_{\phi_0}^{\pi/2}=\frac{\pi}{2}\cdot\frac{10^3}{3}\left[\overbrace{\frac{1}{8}(\underbrace{2\sin^2\phi_0+3}_{2\cdot\frac{9}{10}+3=24/5})\underbrace{\sin\phi_0\cos\phi_0}_{\frac{3}{10}}}^{9/50}+\frac{3}{8}\left(\frac{\pi}{2}-\phi_0\right)\right]$

MAT2500-01/03 21S Test 3 Answers (4)

③ e) f) $V = \frac{\pi}{2} \cdot \frac{10^3}{3} \left[\frac{9}{50} + \frac{3}{8}\left(\frac{\pi}{2} - \arctan 3\right)\right]$

$= \frac{\pi}{2}\left[60 - 125 \arctan(3) + \frac{125}{2}\pi \right]$ Maple agrees!

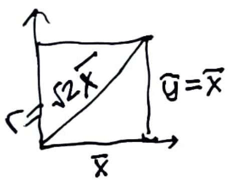
g) $V_{yz} = \int_0^{\frac{\pi}{2}} \int_{\arctan 3}^{\frac{\pi}{2}} \int_0^{10 \sin\phi} \underbrace{((\rho \sin\phi)(\cos\theta))\, \rho^2 \sin\phi}_{\rho^3 \sin^3\phi \cos\theta} \, d\rho\, d\phi\, d\theta \approx 727.67$

$V_{xz} = \int_0^{\frac{\pi}{2}} \int_{\arctan 3}^{\frac{\pi}{2}} \int_0^{10 \sin\phi} \underbrace{((\rho \sin\phi) \sin\theta)\, \rho^2 \sin\phi}_{\rho^3 \sin^3\phi \sin\theta} \, d\rho\, d\phi\, d\theta \approx 727.67$

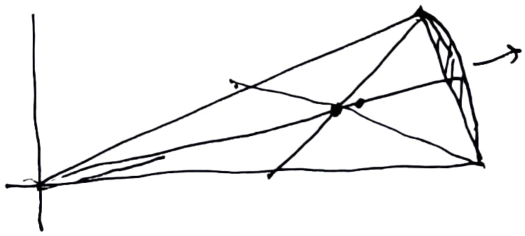
$V_{xy} = \int_0^{\frac{\pi}{2}} \int_{\arctan 3}^{\frac{\pi}{2}} \int_0^{10 \sin\phi} \underbrace{(\rho \cos\phi)\, \rho^2 \sin\phi}_{\rho^3 \sin\phi \cos\phi} \, d\rho\, d\phi\, d\theta \approx 177.37$

$\langle \bar{x}, \bar{y}, \bar{z} \rangle \approx \dfrac{\langle 727.67, 727.67, 177.37 \rangle}{157.42} \approx \langle \underbrace{4.62, 4.62}, \underbrace{1.13}_{\bar{z}} \rangle$

h) $\bar{r} \approx 4.62\sqrt{2} \approx 6.53$ ($\to 6.54$ keeping more digits)



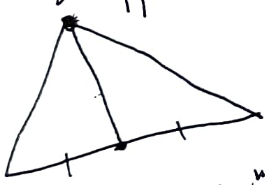
see test sheet diagram:



extra volume pushes centroid along this bisector a bit

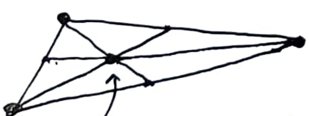
Maple Zoom shows slight downward shift

triangle opp side bisectors



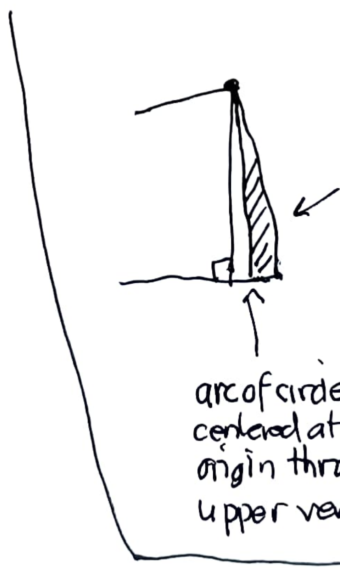
oops "perpendicular" is incorrect.

no one asked for clarification or googled "triangle centroid"



just average coords of 3 vertices

same as center of mass of 3 equal point masses at vertices.



extra volume weights downward very slightly!

arc of circle centered at origin through upper vertex

(see lecture notes 15-4a page 4!)