MAT2500-01/03 21S Test 2 Answers

(1) $f(x,y) = x^2 + y^4 + 2xy$

$f_x(x,y) = 2x + 2y = 0 \rightarrow y = -x$

$f_y(x,y) = 4y^3 + 2x = 0 \rightarrow 4(-x)^3 + 2x = 0$

$-2x^3 + x = 0 \quad x(1-2x^2) = 0$

$x = 0$ or $x = \pm 1/\sqrt{2}$

$\downarrow \qquad\qquad \downarrow$

$y = 0 \qquad y = \mp 1/\sqrt{2}$

critical points:

$(0,0)$, $(1/\sqrt{2}, -1/\sqrt{2})$, $(-1/\sqrt{2}, 1/\sqrt{2})$

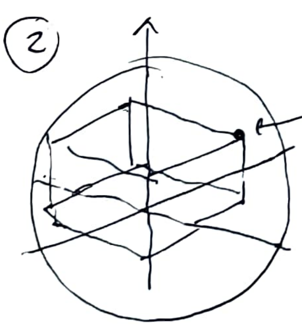
$f_{xx}(x,y) = 2 > 0$ ☺

$f_{yy}(x,y) = 12y^2 \geq 0$ ☺ ?

$f_{xy}(x,y) = 2$

	$(0,0)$	$(1/\sqrt{2}, -1/\sqrt{2})$	$(-1/\sqrt{2}, 1/\sqrt{2})$
f_{xx}	$2 > 0$ ☺	$2 > 0$ ☺ } local min?	ditto only depends on y^2
f_{yy}	0 ?	$6 > 0$ ☺	
f_{xy}	2	2	
$f_{xx}f_{yy} - f_{xy}^2$	$0 - 2^2 < 0$ $\boxed{\text{saddle}}$	$2 \cdot 6 - 2^2 > 0$ confirms $\boxed{\text{local min}}$	$\boxed{\text{local min}}$
$f(x,y)$	$\boxed{0}$	$\frac{1}{2} + \frac{1}{4} + 2\left(-\frac{1}{2}\right) = \frac{3}{4} - 1 = \boxed{-1/4}$	$\boxed{-1/4}$

(2)



$x^2 + y^2 + z^2 = r^2$

(x,y,z) corner in first octant

$x > 0, y > 0, z > 0$

$V = (2x)(2y)(2z)$

$= 8xyz$

[symmetric in (x,y,z) so solution should have $x = y = z$:

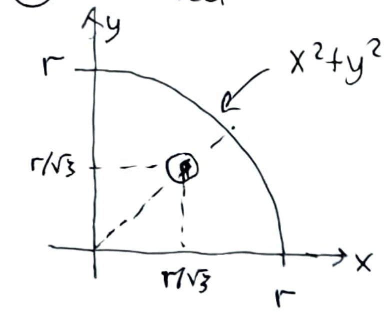
$3x^2 = r^2 \rightarrow x = \frac{r}{\sqrt{3}}$]

solve constraint for z: $\quad z = \sqrt{r^2 - x^2 - y^2} \geq 0$

$V(x,y) = 8xy\sqrt{r^2 - x^2 - y^2}$ (we could maximize V^2 but let's stick with V)

MAT 2500-01/03 21S Test 2 Answers (2)

(2) continued



$x^2+y^2=r^2$

allowed region: interior of quarter circle.

If instead minimize V^2:

$$\mathcal{V} = \frac{V^2}{64} = x^2y^2(r^2-x^2-y^2) = r^2x^2y^2 - x^4y^2 - x^2y^4$$

$$\mathcal{V}_x = 2r^2xy^2 - 4x^3y^2 - 2xy^4 = 2xy^2(r^2-2x^2-y^2) = 0$$

$$\mathcal{V}_y = 2x^2y\underbrace{(r^2-x^2-2y^2)}_{} = 0$$

$2x^2+y^2=r^2$
$x^2+2y^2=r^2$
rest same.

$$V_x = \frac{\partial}{\partial x}\left(8xy(r^2-x^2-y^2)^{1/2}\right)$$
$$= 8y\left[1\cdot(r^2-x^2-y^2)^{1/2} + x\left(\tfrac{1}{2}\right)(\cdots)^{-1/2}(-2x)\right]$$
$$= 8y\left[\frac{r^2-x^2-y^2 \;-\; x^2}{(r^2-x^2-y^2)^{1/2}}\right]$$
$$= \frac{8y\,[r^2-2x^2-y^2]}{(r^2-x^2-y^2)^{1/2}} = 0$$

exchange $x \leftrightarrow y$ (symmetry)

$$V_y = \frac{8x\,[r^2-x^2-2y^2]}{(r^2-x^2-y^2)^{1/2}} = 0$$

$$2\,[2x^2+y^2=r^2] \quad 4x^2+2y^2=2r^2$$
$$x^2+2y^2=r^2 \qquad x^2+2y^2=r^2$$
$$(-)\quad 3x^2+0=r^2$$
$$x^2=r^2/3$$
$$x=r/\sqrt{3}$$

$$2\left(\frac{r^2}{3}\right)+y^2=r^2 \to y^2=r^2/3$$
$$y=r/\sqrt{3}$$

$$z=\sqrt{r^2-\left(\frac{r^2}{3}\right)-\left(\frac{r^2}{3}\right)}=\sqrt{\frac{r^2}{3}}=r/\sqrt{3}$$

so $\boxed{x=y=z=r/\sqrt{3}} \to V = 8\left(\frac{r}{\sqrt{3}}\right)^3 = \boxed{\frac{8}{3\sqrt{3}}r^3}$

The maximum volume of a rectangular box inscribed in a sphere of radius r is $\frac{8}{3\sqrt{3}}r^3$ (a cube of side $2r/\sqrt{3}$).

$$[V_{xx}, V_{yy}, V_{xy}] = \frac{8}{(r^2-x^2-y^2)^{3/2}}\left[-3xy\left(r^2-\frac{2x^2}{3}-y^2\right), -3xy\left(r^2-x^2-\frac{2y^2}{3}\right), r^4+(-3x^2-3y^2)r^2+2x^4+3x^2y^2+2y^4\right]$$

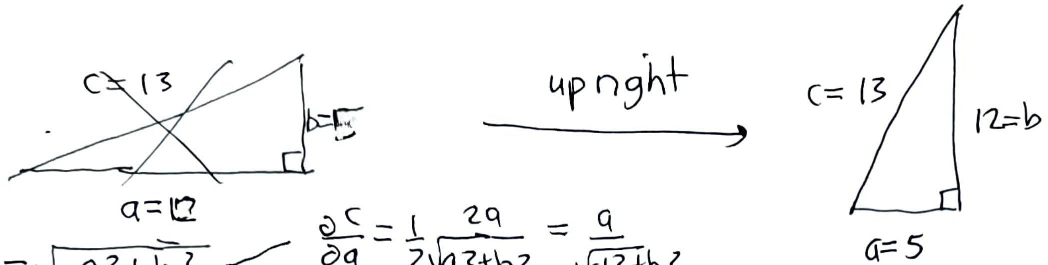
$\downarrow$

$x=y=\frac{r}{\sqrt{3}}$: $\frac{16r}{\sqrt{3}}[-2, -2, -1]$ local max?

$$V_{xx}V_{yy}-V_{xy}^2 = \left(\frac{16r}{\sqrt{3}}\right)^2\Big(\underbrace{(-2)(-2)-(-1)^2}_{4-1=3}\Big) > 0 \text{ confirmed.}$$

MAT 2500-01/03 21S Test 2 Answers (3)

(3)



problem stated sides in order 5 then 12

$$c=\sqrt{a^2+b^2}$$

$$\frac{\partial c}{\partial a}=\frac{1}{2}\frac{2a}{\sqrt{a^2+b^2}}=\frac{a}{\sqrt{a^2+b^2}}$$

$$\frac{\partial c}{\partial b}=\frac{1}{2}\frac{2b}{\sqrt{a^2+b^2}}=\frac{b}{\sqrt{a^2+b^2}}$$

units: meters

$0.2\,\text{cm} = 0.002\,\text{m}$

$|da| \le 0.002$

$|db| \le 0.002$

$|dc| \le ?$

$$dc=\frac{\partial c}{\partial a}da+\frac{\partial c}{\partial b}db=\frac{a\,da}{\sqrt{a^2+b^2}}+\frac{b\,db}{\sqrt{a^2+b^2}}$$

$$|dc|=\frac{|a\,da+b\,db|}{\sqrt{a^2+b^2}}\le\frac{a|da|+b|db|}{\sqrt{a^2+b^2}}$$ triangle inequality

$$=\frac{a(.002)+b(.002)}{\sqrt{a^2+b^2}}=\frac{.002\,(a+b)}{\sqrt{a^2+b^2}}$$

$a=5,\ b=12$: $|da| \le \frac{.002(5+12)}{13}=\left(\frac{17}{13}\right)(.002)$

≈ 0.00261538

≈ 0.0026 m ≈ 0.26 cm (more appropriate units to compare with errors in sides)

$\boxed{\text{The maximum error in the calculated value of the length of the hypotenuse is about 0.26 cm}}$

b) 4 decimal places in cm intended but ambiguous as stated, but even to 4 decimal places in cm, the exact error bars agree with the approximate ones

$c(5.002, 12.002) - c(5,12) = 0.00261543$

$c(5,12) - c(4.998, 11.998) = 0.00261534$

38 in differential approx

error in approximation is in 6th decimal place in cm!

MAT2500-01/03 21S Test 2 Answers (4)

(4) $f(x,y,z) = x^2+2y^2-3z^2 = 3$ level surface

a) $\vec\nabla f(x,y,z) = \langle 2x, 4y, -6z\rangle$

$\vec\nabla f(2,-1,1) = \langle 2(2), 4(-1), -6(1)\rangle = \langle 4,-4,-6\rangle = 2\underbrace{\langle 2,-2,-3\rangle}_{\vec n}$

$\vec r_0 = \langle 2,-1,1\rangle$

$0 = \vec n\cdot(\vec r - \vec r_0) = \langle 2,-2,-3\rangle\cdot\langle x-2, y-(-1), z-1\rangle$

$= 2(x-2) - 2(y+1) - 3(z-1) = 2x-2y-3z \underbrace{-4-2+3}_{-3}$

$\boxed{2x-2y-3z = 3}$

b) $\vec r = \vec r_0 + t\vec n = \langle 2,-1,1\rangle + t\langle 2,-2,-3\rangle$

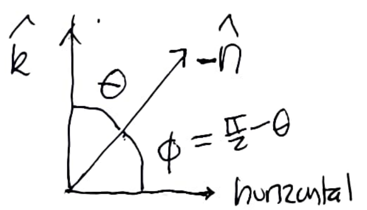
$\boxed{\langle x,y,z\rangle = \langle 2+2t, -1-2t, 1-3t\rangle}$

c) upward unit normal:

$-\hat n = -\dfrac{\langle 2,-2,-3\rangle}{\sqrt{4+4+9}} = \dfrac{\langle -2,2,3\rangle}{\sqrt{17}}$

The idea of learning a topic to understand it is to be able to use the tools to solve a problem you have not already seen done.

It is easy to find the angle wrt to the vertical. One then just has to think about how that angle relates to the horizontal. A simple diagram shows this.

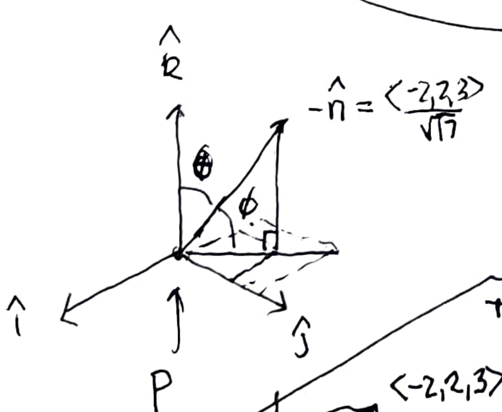


$\cos\theta = -\hat k\cdot\hat n = \langle 0,0,1\rangle\cdot\dfrac{\langle -2,2,3\rangle}{\sqrt{17}} = \dfrac{3}{\sqrt{17}}$

$\theta = \arccos\dfrac{3}{\sqrt{17}} \approx 43.3°$

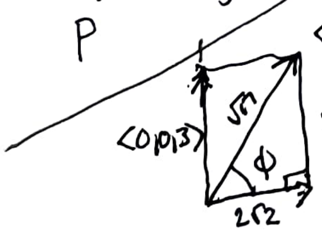
$\boxed{\phi = \frac{\pi}{2} - \arccos\frac{3}{\sqrt{17}} \approx 46.7°}$

angle with horizontal



$\hat\imath\cdot(-\hat n)$, $\hat\jmath\cdot(-\hat n)$ are angles with respect to $\hat\imath$ and $\hat\jmath$ not with respect to the horizontal

the best approach is due to a student!



$\langle -2,2,3\rangle = -\vec n$

project down to horizontal!

$\langle -2,2,0\rangle \equiv \vec h$

$\cos\phi = -\hat n\cdot\hat h = \dfrac{\langle -2,2,3\rangle}{\sqrt{17}}\cdot\dfrac{\langle -2,2,0\rangle}{2\sqrt2}$

$= \dfrac{4+4}{2\sqrt2\sqrt{17}} = \dfrac{2\sqrt2}{\sqrt{17}}$

$\phi = \arccos\left(\dfrac{2\sqrt2}{\sqrt{17}}\right) \approx 46.7°$