MAT2500-01/03 21S Test 1 Answers (1)

a) $\vec{r} = \langle \cos t, \sin t, 2\ln(\cos(\tfrac{t}{2}))\rangle$

This is welldefined for $-\pi < t < \pi$ where $\cos t/2 > 0$.

$\vec{r}(\tfrac{\pi}{2}) = \langle \underbrace{\cos\tfrac{\pi}{2}}_{0}, \underbrace{\sin\tfrac{\pi}{2}}_{1}, 2\ln(\underbrace{\cos\tfrac{\pi}{4}}) \rangle$

$2\ln 2^{-1/2} = 2(-\tfrac{1}{2})\ln 2 = -\ln 2$ $\qquad \tfrac{1}{\sqrt{2}} = 2^{-1/2}$

$\boxed{\vec{r}(\tfrac{\pi}{2}) = \langle 0, 1, -\ln 2\rangle}$

b) $\vec{v} = \vec{r}\,' = \langle -\sin t, \cos t, \dfrac{2\cdot(-\sin\frac{t}{2})(\frac{1}{2})}{\cos t/2}\rangle$

$\boxed{= \langle -\sin t, \cos t, -\tan t/2\rangle}$

$v = \sqrt{\underbrace{\sin^2 t + \cos^2 t}_{1} + \tan^2(t/2)}$ → rest is c) simplification

$= \sqrt{1 + \tan^2(t/2)}$

$= \sqrt{1 + \dfrac{\sin^2 t/2}{\cos^2 t/2}} = \sqrt{\dfrac{\cos^2 t/2 + \sin^2 t/2}{\cos^2 t/2}} = \sqrt{\dfrac{1}{\cos^2 t/2}}$ perfect square

$= \dfrac{1}{|\cos t/2|} = |\sec\tfrac{t}{2}| = \boxed{\sec\tfrac{t}{2}}$

$\vec{v}(\tfrac{\pi}{2}) = \langle -\underbrace{\sin\tfrac{\pi}{2}}_{1}, \underbrace{\cos\tfrac{\pi}{2}}_{0}, -\underbrace{\tan\tfrac{\pi}{4}}_{1}\rangle$

$\boxed{= \langle -1, 0, -1\rangle}$

$\boxed{v(\tfrac{\pi}{2}) = \sqrt{1+1} = \sqrt{2}}$ $\quad (= \sec \pi/4)$ $\quad (= \sqrt{1 + \underbrace{\tan^2 \pi/4}_{1}})$

c) $L = \int_0^{\frac{\pi}{2}} \sec\tfrac{t}{2}\, dt = 2\int_0^{\pi/2} \sec\underbrace{\tfrac{t}{2}}_{u}\, \underbrace{\tfrac{dt}{2}}_{du}$

$\left(\int \sec u\, du = \ln|\sec u + \tan u| + C\right.$

$= 2\ln|\sec(\tfrac{t}{2}) + \tan(\tfrac{t}{2})|\Big|_0^{\frac{\pi}{2}}$

$= 2\ln|\underbrace{\sec\tfrac{\pi}{4}}_{\sqrt{2}} + \underbrace{\tan\tfrac{\pi}{4}}_{1}| = \boxed{2\ln(1+\sqrt{2})}$

$\boxed{\approx 1.7627}$ see next page for arclength function

d) $\hat{T} = \dfrac{\vec{r}\,'}{v} = \dfrac{\langle -\sin t, \cos t, -\tan t/2\rangle}{\sec t/2}$

$\boxed{= \cos t/2 \langle -\sin t, \cos t, -\tan t/2\rangle}$

$\hat{T}(\tfrac{\pi}{2}) = \underbrace{\cos\tfrac{\pi}{4}}_{1/\sqrt{2}} \langle -\underbrace{\sin\tfrac{\pi}{2}}_{1}, \underbrace{\cos\tfrac{\pi}{2}}_{0}, -\underbrace{\tan \pi/4}_{1}\rangle = \boxed{\tfrac{1}{\sqrt{2}}\langle -1, 0, -1\rangle}$

e) $\vec{r}\,'' = \langle -\cos t, -\sin t, -\tfrac{1}{2}\sec^2 t/2\rangle = \vec{a}$

$a = |\vec{r}\,''| = \sqrt{\underbrace{\cos^2 t + \sin^2 t}_{=1} + \tfrac{1}{4}\sec^4 t/2}$

$\vec{r}\,''(\tfrac{\pi}{2}) = \langle -\cos\tfrac{\pi}{2}, -\sin\tfrac{\pi}{2}, -\tfrac{1}{2}\underbrace{\sec^2 \pi/4}_{(\sqrt{2})^2}\rangle$

$= \langle 0, -1, -1\rangle$

$|\vec{r}\,''(\tfrac{\pi}{2})| = \sqrt{1 + \tfrac{1}{4}\underbrace{\sec^4\tfrac{\pi}{4}}_{(\sqrt{2})^4 = 4}} = \sqrt{2}$

f) $\vec{r} = \vec{r}(\tfrac{\pi}{2}) + s\,\vec{r}\,'(\tfrac{\pi}{2})$

$\langle x, y, z\rangle = \langle 0, 1, -\ln 2\rangle + s\langle -1, 0, -1\rangle$

$\boxed{= \langle -s, 1, -\ln 2 - s\rangle}$

g) $\vec{b}(\tfrac{\pi}{2}) = \vec{r}\,'(\tfrac{\pi}{2}) \times \vec{r}\,''(\tfrac{\pi}{2})$

$= \langle -1, 0, -1\rangle \times \langle 0, -1, -1\rangle$

$\overset{\text{maple}}{=} \langle -1, -1, 1\rangle$

$\hat{B}(\tfrac{\pi}{2}) = \dfrac{\langle -1, -1, 1\rangle}{\sqrt{3}}$

h) $\hat{N}(\tfrac{\pi}{2}) = \hat{B}(\tfrac{\pi}{2}) \times \hat{T}(\tfrac{\pi}{2})$

$= \dfrac{\langle -1, -1, 1\rangle}{\sqrt{3}} \times \dfrac{1}{\sqrt{2}}\langle -1, 0, -1\rangle$

$\overset{\text{maple}}{=} \boxed{\tfrac{1}{\sqrt{6}}\langle 1, -2, -1\rangle}$

i) $\vec{n} \propto \vec{b} \propto \hat{B}$ take $\vec{n} = \langle +1, 1, -1\rangle$ simpler!

$\vec{r}_0 = \langle 0, 1, -\ln 2\rangle$

$0 = \vec{n}\cdot(\vec{r} - \vec{r}_0) = \langle 1, 1, -1\rangle \cdot \langle x-0, y-1, z+\ln 2\rangle$

$= x + (y-1) - (z + \ln 2) = 0$

$\boxed{x + y - z = 1 + \ln 2}$

j) $\kappa(\tfrac{\pi}{2}) = \dfrac{|\vec{b}(\pi/2)|}{v(\pi/2)^3} = \dfrac{\sqrt{3}}{(\sqrt{2})^3} = \boxed{\dfrac{\sqrt{3}}{2\sqrt{2}}}$

$\boxed{= \dfrac{\sqrt{6}}{4}}$

$\rho(\tfrac{\pi}{2}) = \boxed{\dfrac{4}{\sqrt{6}}} = \dfrac{4}{6}\sqrt{6} = \boxed{\dfrac{2\sqrt{6}}{3}}$

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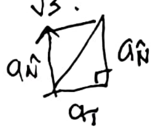
k) $a_{\hat{T}}(\frac{\pi}{2}) = \hat{T}(\frac{\pi}{2}) \cdot \vec{a}(\frac{\pi}{2}) = \frac{1}{\sqrt{2}}\langle -1,0,-1\rangle \cdot \langle 0,-1,-1\rangle = \frac{1}{\sqrt{2}}(0+0+1) = \boxed{\frac{1}{\sqrt{2}}}$

$a_{\hat{N}}(\frac{\pi}{2}) = \hat{N}(\frac{\pi}{2}) \cdot \vec{a}(\frac{\pi}{2}) = \frac{1}{\sqrt{6}}\langle 1,-2,-1\rangle \cdot \langle 0,-1,-1\rangle = \frac{1}{\sqrt{6}}(0+2+1) = \boxed{\frac{\sqrt{3}}{\sqrt{2}}}$

$a_{\hat{T}}(\frac{\pi}{2})^2 + a_{\hat{N}}(\frac{\pi}{2})^2 = \sqrt{\frac{1}{2}+\frac{3}{2}} = \sqrt{2} = |\vec{a}(\frac{\pi}{2})|$ ✓

↑

note: ratio $\sqrt{3}$!



l) $\vec{c}(\frac{\pi}{2}) = \vec{r}(\frac{\pi}{2}) + \rho(\frac{\pi}{2})\hat{N}(\frac{\pi}{2})$

$= \langle 0,1,-\ln 2\rangle + \frac{2\cancel{\sqrt{6}}}{3}\frac{\langle 1,-2,-1\rangle}{\cancel{\sqrt{6}}}$

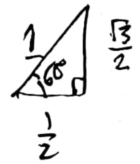
$= \langle 0,1,-\ln 2\rangle + \langle \frac{2}{3}, -\frac{4}{3}, -\frac{2}{3}\rangle$

$\boxed{= \langle \frac{2}{3}, -\frac{1}{3}, -\ln 2 - \frac{2}{3}\rangle}$

m) $\cos\theta = \hat{T}(\frac{\pi}{2}) \cdot \hat{a}(\frac{\pi}{2}) = \frac{1}{\sqrt{2}}\langle -1,0,-1\rangle \cdot \frac{\langle 0,-1,-1\rangle}{\sqrt{2}}$

$= \frac{1}{2}(0+0+1) = \frac{1}{2}$

$\theta = \arccos\frac{1}{2} = \boxed{\frac{\pi}{3} = 60^\circ}$ exactly



oops, I forgot this on page 1!

c) arclength function:

$\boxed{s} = \int_0^t \sec\frac{u}{2}\,du = 2\ln|\sec\frac{u}{2} + \tan\frac{u}{2}|\,\Big|_0^t$

$= 2\ln|\sec\frac{t}{2} + \tan\frac{t}{2}| - \underbrace{2\ln|\underbrace{\sec 0}_{1} + \underbrace{\tan 0}_{0}|}_{0}$

$\boxed{= 2\ln|\sec\frac{t}{2} + \tan\frac{t}{2}|}$