MAT1505-01/02 21F Quiz 9 Print Name (Last, First) ________________ , ________________

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation when appropriate). Indicate where technology is used and what type (Maple, GC). Maple may not substitute for any hand calculations unless explicitly stated, but use it to check each step if you want to be safe.

1. Consider the unusual function $f(x) = \dfrac{\sin(\tan(x)) - \tan(\sin(x))}{\arcsin(\arctan(x)) - \arctan(\arcsin(x))} = \dfrac{Numer(x)}{Denom(x)}$.

a) Plot $f(x)$ for $x = -1..1$, **then** $x = -0.1..0.1$. What does the limit appear to be from the first plot? Clearly the second plot shows something unusual is happening near the origin.

b) Use the taylor command [> taylor(g(x),x=0,12)] for $Numer(x)$, $Denom(x)$ and write down the two results.

c) Looking at these, what can you conclude about $\lim_{x \to 0}(f(x))$?

[Hint: Divide the numerator and denominator series by the lowest power.]

d) Evaluate $\lim_{x \to 0}(f(x))$ with Maple to confirm.

e) Confirm this by using the above taylor command directly on $f(x)$. Why does this explain the limit?

2. The error function is defined by $\operatorname{erf}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^x e^{-t^2}\,dt = \dfrac{2}{\sqrt{\pi}} \int e^{-x^2}\,dx$ (*ignore constants of integration*)

where the last equality follows from $\operatorname{erf}(0) = 0$.

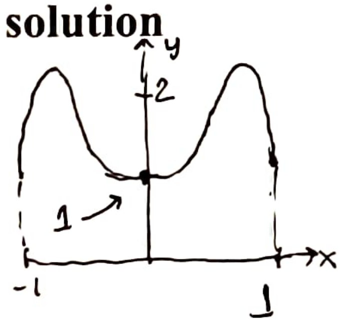
a) Find the Taylor series for this function by using the exponential power series with $x \to -x^2$ and integrate term by term.

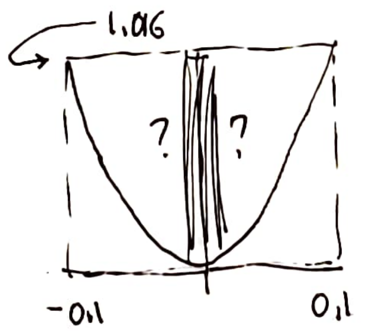
b) Evaluate the first two derivatives of e^{-x^2} at $x = 0$ to find the first two nonzero terms in the power series for e^{-x^2}. Compare with the taylor command for the second Taylor polynomial. Dare you try to get the next nonzero term using this approach?

c) Use the absolute convergence ratio test to confirm that the error function power series has an infinite radius of convergence (which follows from the result for the exponential power series itself has an infinite radius of convergence which extends to those power series obtained from it by "tricks").

▶ **solution**

① a)





The plot clearly shows $\lim_{x \to 0} y = 1$ but wierd numeric problems happen close to $x=0$.

b) taylor(Numer(x)) $= -\frac{1}{30}x^7 - \frac{29}{756}x^9 + \cdots$

taylor(Denom(x)) $= -\frac{1}{30}x^7 + \frac{13}{756}x^9 - \cdots$

c) same lowest powers so ratio approaches 1 (only lowest powers matter)

$$\lim_{x \to 0} \frac{-\frac{1}{30}x^7 - \frac{29}{756}x^9 \cdots / -\frac{1}{30}x^7}{-\frac{1}{30}x^7 + \frac{13}{756}x^9 \cdots / -\frac{1}{30}x^7} = \lim_{x \to 0} \frac{1 + \frac{30 \cdot 29}{756}x^2 + \cdots}{1 - \frac{13 \cdot 30}{756}x^2 + \cdots} = \frac{1}{1} = 1$$

c) $\lim_{x \to 0} f(x) = 1$ Maple confirms

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① e) taylor $(f(x), x=0, 6) = \underbrace{1}_{} + \frac{5}{3}x^2 + \frac{1313}{1890}x^4 + O(x^6)$

constant term is limit as $x\to 0$ since all higher order terms vanish in this limit

② a) $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int e^{-x^2}\,dx$

$= \frac{2}{\sqrt{\pi}} \int \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}\,dx$

$= \sum_{n=0}^{\infty} \frac{2}{\sqrt{\pi}} (-1)^n \frac{x^{2n}}{n!}\,dx$

$= \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{2}{\sqrt{\pi}} \frac{x^{2n+1}}{n!(2n+1)}}$

$\operatorname{erf}(x) = \int_0^x e^{-t^2}\,dt$ $\quad \operatorname{erf}(0) = 0$

$\operatorname{erf}'(0) = \frac{2}{\sqrt{\pi}}$

$\operatorname{erf}''(0) = 0$

$\operatorname{erf}'''(0) = -\frac{4}{\sqrt{\pi}}$

b) $\operatorname{erf}'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}$

$\operatorname{erf}''(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}(-2x) =$

$\operatorname{erf}'''(x) = \frac{2}{\sqrt{\pi}}\left(e^{-x^2}(-2x)^2 + e^{-x^2}(-2)\right) = \frac{2}{\sqrt{\pi}} e^{-x^2}(4x^2-2)$

$\operatorname{erf}(x)$: $T_3(x) = \frac{2}{\sqrt{\pi}}\left(\frac{x}{1!} - 2\frac{x^3}{3!} + \right) = \frac{2}{\sqrt{\pi}}\left(x - \frac{1}{3}x^3\right)$

$f(x) = e^{-x^2}$ $\quad f(0) = 1$

$f'(x) = e^{-x^2}(-2x)$ $\quad f'(0) = 0$

$f''(x) = e^{-x^2}(-2x)^2 + e^{-x^2}(-2) = e^{-x^2}(4x^2-2)$ $\quad f''(0) = -2$

$e^{-x^2} = 1 - \frac{2}{2!}x^2 + \cdots = \boxed{1 - x^2 + \cdots}$ (Maple agrees)

next nonzero term requires 2 more derivatives, check out Maple worksheet. Laborious by hand!

c) $\left|\frac{a_{n+1}}{a_n}\right| = \dfrac{\frac{2}{\sqrt{\pi}} \frac{|x|^{2(n+1)+1}}{(n+1)!}}{\frac{2}{\sqrt{\pi}} \frac{|x|^{2n+1}}{n!}} = \frac{n!}{(n+1)!}|x|^2 = \frac{\cancel{n!}}{(n+1)\cancel{n!}}|x|^2 = \frac{|x|^2}{n+1} \longrightarrow 0$ as $n\to\infty$

< 1 for all values of x so converges everywhere.