MAT1505-01/02 21F Quiz 6 Print Name (Last, First) ________________ , ______________|___

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation when appropriate). Indicate where technology is used and what type (Maple, GC). Maple may not substitute for any hand calculations unless explicitly stated, but use it to check each step if you want to be safe.

1. a) Find the arclength function $s(x)$ for the curve $y = f(x) = x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}} = \frac{x^{\frac{1}{2}}(3-x)}{3}$ with $x = 0$ as the reference point, for $x \geq 0$.

Note that the integrand simplifies because it involves a perfect square. However, this function has a vertical tangent at the origin, so we must define the improper integral for the arclength from this reference point using a limit.

b) Compare your result to the factored form of the original function. Notice the similarity. These perfect square functions seem to have this simple property.

c) Evaluate the arclength between 0 and 1, exactly and approximately to 4 decimal places.

d) Compare your result with the length of the secant line between the endpoints.

2. Consider the curve $y = \ln(\sin(x))$, $0 < x < \pi$.

a) Find the arclength function $s(x)$ zeroed at $x = \frac{\pi}{2}$ by using the trig identity $1 + \cot^2(x) = \csc^2(x)$ to remove the radical by the perfect square trick. Use Maple to give you the trig antiderivative you need for this integral. Warning: Maple will not evaluate the definite integral in this case for reasons that are not clear.

b) Find the arclength over the interval $\frac{\pi}{3} \leq x \leq \frac{2\pi}{3}$ using this arclength function.

[If you wish, you can check this using the $n = 2$ secant line approximation to the arclength. The arclength is about 1 percent longer.]

3. In place of either 1 or 2 you can consider the section 8.1 arclength contest in the textbook described after the section. To get started you could consider the nonnegative function on the unit interval which vanishes at the endpoints to be a parabola $y = a\,x\cdot(1-x)$ choosing the coefficient to be the reciprocal of the integral from 0 to 1 of the rest of the expression so that it has unit area. You can compare this with another such function $y = a\sin(\pi x)$ again choosing the normalizing coefficient in the same way. Which one gives a shorter arclength? In both cases, you can just do a numerical integration to evaluate the decimal value directly although Maple can give exact values for the length.

1 a) $s(x) = \int_0^x \sqrt{1 + f'(t)^2}\,dt = \int_0^x \sqrt{\left(\frac{1}{2}t^{-1/2} + \frac{1}{2}t^{1/2}\right)^2}\,dt = \int_0^x \underbrace{\frac{1}{2}t^{-1/2}}_{\text{undefined!}} + \frac{1}{2}t^{1/2}\,dt$

$f(x) = x^{1/2} - \frac{1}{3}x^{3/2}$

$f'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{3}\left(\frac{3}{2}\right)x^{1/2} = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{1/2}$

$1 + f'(x)^2 = 1 + \left(\frac{1}{2}x^{-1/2} - \frac{1}{2}x^{1/2}\right)^2$

$= 1 + \left(\frac{1}{4x} - \frac{2}{2\cdot 2}x^{-1/2}x^{1/2} + \frac{1}{4}x\right)$

$= 1 + \left(\frac{1}{4x} - \frac{1}{2} + \frac{1}{4}x\right)$

$= \left(\frac{1}{4x} + \frac{1}{2} + \frac{1}{4}x\right)$

$= \left(\frac{1}{2}x^{-1/2} + \frac{1}{2}x^{1/2}\right)^2$

$= \lim_{z\to 0^+} \int_z^x \frac{1}{2}t^{-1/2} + \frac{1}{2}t^{1/2}\,dt$

$= \lim_{z\to 0^+} \left(\frac{1}{2}\frac{t^{1/2}}{1/2} + \frac{1}{2}\frac{t^{3/2}}{3/2}\right)\Big|_z^x$

$= \lim_{z\to 0^+} \left(t^{1/2} + \frac{1}{3}t^{3/2}\right)\Big|_z^x$

$= \lim_{z\to 0^+} \left(x^{1/2} + \frac{1}{3}x^{3/2} - \underbrace{z^{1/2}}_{0} - \underbrace{\frac{1}{3}z^{3/2}}_{0}\right)$

$= \boxed{x^{1/2} + \frac{1}{3}x^{3/2} = s(x)}$

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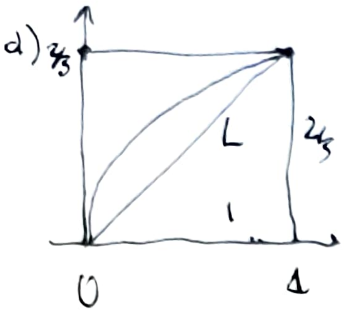
① b) $S(x) = x^{1/2} + \frac{1}{3}x^{3/2}$

$= x^{1/2}\left(1 + \frac{x}{3}\right) = \frac{(3+x)}{3}x^{1/2} \leftrightarrow f(x) = x^{1/2}\frac{(3-x)}{3}$

just a change of sign!

c) $S(1) = 1^{1/2}\frac{(3+1)}{3} = \boxed{\frac{4}{3}} \approx 1.33$

d)



$L = \sqrt{1^2 + (2/3)^2}$

$= \sqrt{1 + \frac{4}{9}} = \sqrt{\frac{13}{9}}$

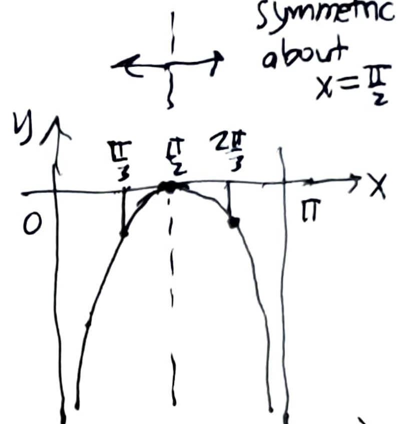
$= \frac{\sqrt{13}}{3} \approx 1.20$

$\boxed{S(1) \text{ is a bit more than } L}$

$f(1) = 1\frac{(3-1)}{3} = \frac{2}{3}$

② $y = \ln(\sin x) \quad 0 < x < \pi$

symmetric about $x = \frac{\pi}{2}$



a) $\frac{dy}{dx} = \frac{1}{\sin x}(\cos x) = \cot x \to \pm\infty$

as $x \to 0, \pi$

$\sin x \to 0^+$

$\ln s \to -\infty$

$x = \frac{\pi}{2} \to y = \ln(\sin\frac{\pi}{2})$

$= \ln 1 = 0$

arclength $\frac{\pi}{2} \to \frac{2\pi}{3}$

= arclength $\frac{\pi}{3} \to \frac{\pi}{2}$

so double first

$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \cot^2 x = \csc^2 x \to \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \csc^2 x = \frac{1}{\sin^2 x}$

$S(x) = \int_{\pi/2}^{x} \csc t \, dt$

> 0 for $0 < t < \pi$

where $\sin x > 0$

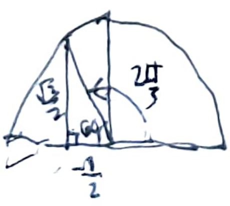
$\int \csc x \, dx = -\ln(\csc x + \cot x)$

$= -\ln(\csc t + \cot t)\Big|_{\pi/2}^{x} = -\ln(\csc x + \cot x) + \underbrace{\ln(\overset{1}{\csc\frac{\pi}{2}} + \overset{0}{\cot\frac{\pi}{2}})}_{0}$

$= \boxed{-\ln(\csc x + \cot x)}$

b) $S\left(\frac{2\pi}{3}\right) = -\ln\left(\csc\frac{2\pi}{3} + \cot\frac{2\pi}{3}\right) = -\ln\left(\overbrace{\frac{1}{\sin\frac{2\pi}{3}} + \frac{\cos 2\pi/3}{\sin 2\pi/3}}^{\frac{1}{\sqrt{3}}}\right) = -\ln\left(\frac{1}{\sqrt{3}}\right) = \ln\sqrt{3} = \frac{1}{2}\ln 3$

$1/(\sqrt{3}/2) \qquad \frac{-1/2}{\sqrt{3}/2}$



$L = 2S\left(\frac{2\pi}{3}\right) = 2\left(\frac{1}{2}\ln 3\right) = \boxed{\ln 3 \approx 1.0987}$

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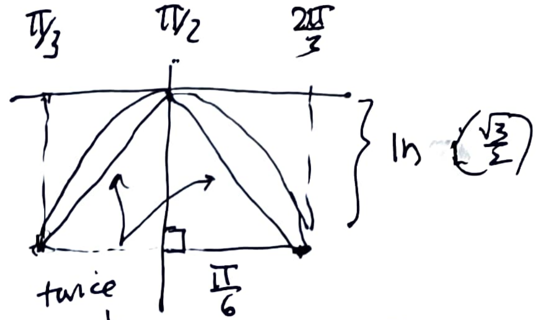
② continued.

$f(x) = \ln \sin x$

$f\left(\frac{2\pi}{3}\right) = \ln \sin\frac{2\pi}{3} = \ln\left(\frac{\sqrt{3}}{2}\right)$



$L_{sec} = 2\sqrt{\left(\frac{\pi}{6}\right)^2 + \ln\left(\frac{\sqrt{3}}{2}\right)^2}$

≈ 1.08996

$\frac{L}{L_{sec}} \approx \frac{1.0987}{1.0860} \approx 1.012$

about 1% larger

(bob was careless in Maple with an extra factor of 2)

③ $y = a\,x(1-x)$

$1 = \int_0^1 a\,x(1-x)\,dx = a\int_0^1 x - x^2\,dx = a\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = a\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{a}{6} \to a = 6$

$f(x) = 6x(1-x) = 6(x - x^2)$ $\quad f'(x) = 6(1-2x)$ $\quad f'(x)^2 + 1 = 1 + 36(1 - 4x + 4x^2)$

$= 37 - 144x + 144x^2$

$L_{parabola} = \int_0^1 \sqrt{37 + 144x^2 - 144x}\,dx \underset{\text{Maple}}{=} \frac{\sqrt{37}}{2} + \frac{\operatorname{arcsinh} 6}{12} \approx 3.249$

$y = a\sin(\pi x)$

$1 = \int_0^1 a\sin(\pi x)\,dx = -a\cos\pi x\Big|_0^1 = a(-(-1)+1) = 2a \to a = \frac{1}{2}$

$g(x) = \frac{1}{2}\sin\pi x$ $\quad g'(x) = \frac{\pi}{2}\cos\pi x$; $\quad 1 + g'(x)^2 = 1 + \frac{\pi^2}{4}\cos^2\pi x$

$L_{sine} = \int_0^1 \sqrt{1 + \frac{\pi^2}{4}\cos^2\pi x} \underset{\text{Maple}}{=} \frac{\sqrt{4+\pi^4}}{\pi}\,\text{EllipticE}\left(\frac{\pi^2}{\sqrt{\pi^4+4}}\right) \approx 3.366$

so $L_{parabola} < L_{sine}$ parabola beats sine!