MAT1505-01/02 21F Quiz 5 Print Name (Last, First) ____________, ____________|____

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation when appropriate). Indicate where technology is used and what type (Maple, GC), but remember that Maple is for checking hand calculations, not substituting for them.

You may use technology to evaluate the necessary antiderivatives, but you must use the limit definition which allows evaluation of the improper integrals. Check your work by directly evaluating these integrals with technology.

1. $\displaystyle\int_0^\infty \frac{x}{(x^2+a^2)^{\frac{3}{2}}}\,dx$, $a > 0$ Either use u-substitution to find the antiderivative and then continue with the improper integral, or use it to transform the integral to a new improper integral and continue.

2. $\displaystyle\int_2^\infty \frac{1}{x\sqrt{x^2-4}}\,dx$ a) Use Maple to find the antiderivative and confirm by hand differentiation that the result simplifies to the integrand.
b) Use that antiderivative to evaluate the improper integral. Be sure to rewrite this as a sum of two definite integrals to deal with both the explicit infinity and implicit infinity involved here.

▶ **solution**

① [Notice that for very large x: $\dfrac{x}{(x^2+a^2)^{3/2}} \sim \dfrac{x}{(x^2)^{3/2}} \sim \dfrac{x}{x^3} \sim \dfrac{1}{x^2}$ which integrates to $-x^{-1} \to 0$ as $x\to\infty$
so we expect this to converge.]

$$\int \frac{x\,dx}{(x^2+a^2)^{3/2}} = \int u^{-3/2}\frac{du}{2} = \frac{1}{2}\int u^{-3/2}du = \frac{1}{2}\frac{u^{-1/2}}{-1/2} = -(x^2+a^2)^{-1/2} = -\frac{1}{(x^2+a^2)^{1/2}}$$

[$u = x^2+a^2$
$du = 2x\,dx$
$\frac{1}{2}du = x\,dx$]

$$\int_0^\infty \frac{x\,dx}{(x^2+a^2)^{3/2}} = \lim_{t\to\infty}\int_0^t \frac{x\,dx}{(x^2+a^2)^{3/2}} = \lim_{t\to\infty} -\frac{1}{(x^2+a^2)^{1/2}}\Big|_0^t$$

$$= \lim_{t\to\infty}\Big[\underbrace{-\frac{1}{(t^2+a^2)^{1/2}}}_{\to 0} + \frac{1}{(a^2)^{1/2}}\Big] = \boxed{\frac{1}{a}}$$

② $$\int_2^\infty \frac{dx}{x\sqrt{x^2-4}} = \int_2^3 \frac{dx}{x\sqrt{x^2-4}} + \int_3^\infty \frac{dx}{x\sqrt{x^2-4}}$$

$$= \lim_{t\to 2^+}\int_t^3 \frac{dx}{x\sqrt{x^2-4}} + \lim_{t\to\infty}\int_3^t \frac{dx}{x\sqrt{x^2-4}}$$

[Remark: near $x\to 2^+$: $\dfrac{1}{x\sqrt{(x-2)(x+2)}} \sim \dfrac{1}{2\sqrt{x-2}\sqrt{4}} \sim (x-2)^{-1/2}$
integrates to $(x-2)^{1/2} \to 0$!
near $x\to\infty$: $\dfrac{1}{x\sqrt{x^2-4}} \sim \dfrac{1}{x\cdot x} = \dfrac{1}{x^2}$
integrates to $-1/x \to 0$ as $x\to\infty$
so we expect convergence at both ends]

3 is arbitrary ($2<3$!) but the simplest integer to use in dividing up the integral

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(2) $\int \frac{dx}{x\sqrt{x^2-4}} \underset{\text{Maple}}{=} -\frac{1}{2}\arctan\left(\frac{2}{\sqrt{x^2-4}}\right) + C$

$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$

$\frac{d}{dx}\left(-\frac{1}{2}\arctan\left(2(x^2-4)^{-1/2}\right)\right)$

$= -\frac{1}{2}\,\frac{1}{1+\left(\frac{4}{x^2-4}\right)}\;\underbrace{\underbrace{\frac{d}{dx}\left(2(x^2-4)^{-1/2}\right)}_{2\left(-\frac{1}{2}\right)(x^2-4)^{-3/2}(2x)}}_{\frac{-2x}{(x^2-4)^{3/2}}}$

$= \frac{x}{1+\frac{4}{x^2-4}}\cdot\frac{1}{(x^2-4)\sqrt{x^2-4}} = \frac{x}{\sqrt{x^2-4}}\;\frac{1}{\underbrace{\cancel{(x^2-4)}\cancel{+4}}_{\frac{1}{x^2}}} = \frac{1}{x\sqrt{x^2-4}}$ ✓

yay!

$\int_2^\infty \frac{dx}{x\sqrt{x^2-4}} = \lim_{t\to 2^+}\left(-\frac{1}{2}\arctan\left(\frac{2}{\sqrt{x^2-4}}\right)\right)\Big|_t^3 + \lim_{t\to\infty}\left(-\frac{1}{2}\arctan\left(\frac{2}{\sqrt{x^2-4}}\right)\right)\Big|_3^t$

$= \lim_{t\to 2^+}\left[-\frac{1}{2}\arctan\left(\frac{2}{\sqrt{5}}\right) + \underbrace{\frac{1}{2}\arctan\underbrace{\frac{2}{\sqrt{t^2-4}}}_{\to\infty}}_{\to\pi/2}\right] + \lim_{t\to\infty}\left[\underbrace{-\frac{1}{2}\arctan\left(\underbrace{\frac{2}{\sqrt{t^2-4}}}_{\to 0}\right)}_{\to 0} + \frac{1}{2}\arctan\frac{2}{\sqrt{5}}\right]$

can cancel since both limits finite

$= \boxed{\frac{\pi}{4}}$