MAT1505-01/02 21F Quiz 4 Print Name (Last, First) __________________, __________________

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). INDICATE where technology is used and what type (Maple, GC).

1. a) Use the relation $\dfrac{x^2}{1+x^2} = \dfrac{x^2+1-1}{1+x^2} = 1 - \dfrac{1}{1+x^2}$ (cheap trick polynomial long division!) to complete the integration by parts of $\int x \arctan(x)\, dx$, showing all steps.

b) Use that result to evaluate the area between the two graphs $y = x \arctan\left(\dfrac{x}{a}\right)$, $y = x\arctan(1)$ by doing an obvious variable substitution first to the area integral and then using the result from part a) for the antiderivative. Support your initial integral formula for the area with an explanatory diagram. Does your result make sense given your plot? Can you compare with the rectangle containing the two curves?

c) What is the average value of $x \arctan\left(\dfrac{x}{a}\right)$ over $0 \le x \le a$?

2. Evaluate $\int_0^{\pi} x \sin(x) \cos(x)\, dx$ using integration by parts, step by step.

① a) $\int x \underbrace{\arctan x}_{u}\, \underbrace{dx}_{} = \underbrace{\frac{x^2}{2}}_{v} \underbrace{\arctan x}_{u} - \int \underbrace{\frac{x^2}{2}}_{v} \underbrace{\frac{1}{1+x^2}dx}_{du} = \frac{x^2}{2}\arctan x - \frac{1}{2}\underbrace{\int \frac{x^2}{1+x^2}dx}$

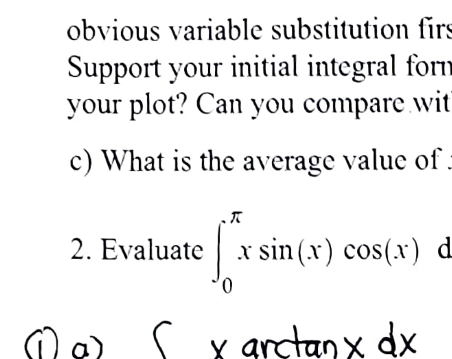
$u = \arctan x \qquad dv = x\,dx$

$du = \frac{1}{1+x^2}dx \qquad v = \frac{x^2}{2}$

$= \int 1 - \frac{1}{1+x^2}dx = x - \arctan x$

$= \frac{x^2}{2}\arctan x - \frac{1}{2}(x - \arctan x) + C$

$= \boxed{\frac{x^2+1}{2}\arctan x - \frac{1}{2}x + C} = \frac{x^2}{2}\arctan x + \frac{1}{2}\arctan x - \frac{1}{2}x + C$



$A = \int_{0 \to 0}^{a \to 1} \left(x \arctan 1 - x \arctan \underbrace{\frac{x}{a}}\right) dx$ ($x \to au$, $dx \to a\,du$)

$u = \frac{x}{a},\ du = \frac{dx}{a}$

$x = 0 \to u = 0$

$x = a \to u = 1$

$= a^2 \int_0^1 (u \arctan 1 - u \arctan u)\, du$

$= a^2 \left[\frac{u^2}{2}\arctan 1 \Big|_0^1 - \left(\frac{u^2+1}{2}\arctan u - \frac{1}{2}u \right)\Big|_0^1 \right]$ π/4!

$= a^2\left[\frac{1}{2}\arctan 1 - \left(\arctan 1 - \frac{1}{2}\right) + \cancel{\frac{1}{2}\arctan(0)} - \cancel{\frac{1}{2}0} \right] = a^2\left(\frac{1}{2} - \frac{1}{2}\arctan 1\right)$

$= \boxed{a^2\left(\frac{1}{2} - \frac{\pi}{8}\right)}$

$\approx 0.107 > 0$ ✓

$A_{rectangle} = a\left(a\frac{\pi}{4}\right) = \frac{\pi}{4}a^2$

$\frac{A}{A_{rectangle}} = \frac{a^2\left(\frac{1}{2} - \frac{\pi}{8}\right)}{a^2(\pi/4)} = \frac{4}{\pi}\left(\frac{1}{2} - \frac{\pi}{8}\right) = \frac{2}{\pi} - \frac{1}{2} \approx 0.14$ compare with plot for $a = 1$

about half a quarter of the area, so in the right ballpark.

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(2) $$\int_0^{\pi} \underbrace{x}_{u}\, \underbrace{\sin x \cos x \, dx}_{dv} = \frac{x}{2}\sin^2 x \Big|_0^{\pi} - \underbrace{\int_0^{\pi} \frac{1}{2}\sin^2 x \, dx}$$

$u = x$ $\quad dv = \sin x \cos x\, dx$

$du = dx$ $\quad v = \int \underbrace{(\sin x)}_{w} \underbrace{(\cos x\, dx)}_{dw}$

$$= \frac{1}{2}w^2 = \frac{1}{2}\sin^2 x$$

or jump using Maple

OR $\int_0^{\pi} \frac{1}{2}\, \frac{1}{2}(1 - \cos 2x)\, dx$

$\frac{1}{4}\left(x - \frac{1}{2}\sin 2x\right)\Big|_0^{\pi}$

OR

$\frac{1}{4}(x - \sin x \cos x)\Big|_0^{\pi}$

$$= \left(\frac{x}{2} - \frac{1}{8}\right)\sin 2x - \frac{x}{4} \Big|_0^{\pi}$$

$$= \left(\frac{\pi}{2} - \frac{1}{8}\right)\underbrace{\sin 2\pi}_{0} - \frac{\pi}{4} - \left(0 - \frac{1}{8}\sin 0\right) + \frac{0}{4}$$

$$= \boxed{-\frac{\pi}{4}}$$

Because of trig identities there are many variations on working this problem but the antiderivatives all differ by a constant since $\sin^2 x = 1 - \cos^2 x$ etc.

OR $\int \cos x\, (\sin x\, dx)$

$= \cdots = -\frac{1}{2}\cos^2 x + C_2$

(1) b) $A_{avg} = \frac{1}{a}\int_0^a x \arctan\left(\frac{x}{a}\right) dx = \frac{1}{a}\left(a^2\left(\frac{\pi}{4} - \frac{1}{2}\right)\right) = \boxed{a\left(\frac{\pi}{4} - \frac{1}{2}\right)}$

$\approx 0.29a \quad > 0$ ✓