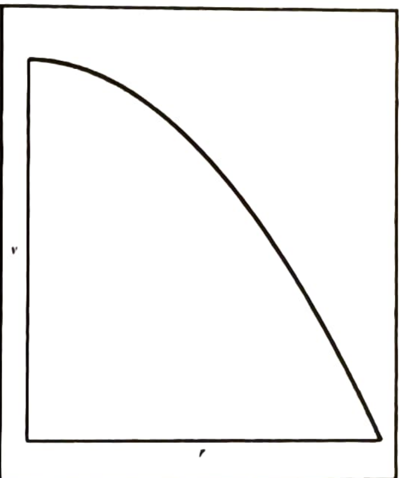
MAT1505-01/02 21F Quiz 1 Print Name (Last, First) ____________________

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). INDICATE where technology is used and what type (Maple, GC).

Given the velocity profile $v(r) = \dfrac{P}{4\,\eta\,\ell}\left(R^2 - r^2\right)$ for $0 \le r \le R$ for blood flow in a long cylindrical tube as a function of the radial distance r from the central axis of the tube of radius R, where $P > 0$ is the pressure (more pressure faster flow) $\eta > 0$ is the blood viscosity (thicker blood slower flow) and $\ell > 0$ is the length of the tube (idealized artery! longer tube, slower flow?). This is fast in the center and tapers off to zero at the outer edge. [See diagram.]



A more appropriate average velocity of the blood flow would be one weighted by the differential area $dA = 2\,\pi\, r\, dr$ represented by the annular strip of thickness dr, to give more weight to the larger ring of blood flowing farther from the center, namely

$$\langle v\rangle = \frac{\int_0^R v(r)\,2\,\pi\, r\,dr}{\int_0^R 2\,\pi\, r\,dr} = \frac{\int_0^R v(r)\,2\,\pi\, r\,dr}{\pi R^2}.$$ [This line is all you need.]

[One could roughly approximate the blood flow as a uniform velocity flow at this average velocity independent of radius. It would give the same total amount of blood passing any cross-section at any given moment. We will come back to this later in the course.]

a) For the given velocity profile, introduce the change of variable $x = \dfrac{r}{R}$ or $r = x\,R$ into this integral expression to re-express it completely in terms of the new dimensionless variable x and re-express the coefficient of the resulting integral (don't evaluate the integral yet!) for the average velocity $\langle v\rangle$ in terms of the maximum velocity $v_{max} = v(0)$.

b) Now integrate by hand the resulting integral in the variable x.

c) At what fraction of the radius R does the actual velocity equal this average velocity $\langle v\rangle$?

d) Check your evaluation in part b) using Maple. Does it agree with your hand result as it should?

a) $\langle v\rangle = \dfrac{\int_0^R \frac{P}{4\eta\ell}(R^2-r^2)(2\pi r)\,dr}{\pi R^2} = \dfrac{P(2\pi)}{4\pi\eta\ell}\int_0^R \dfrac{R^2-r^2}{R^2}\, r\,dr = \dfrac{P}{2\eta\ell}\int_{0 \to 0}^{R \to 1}\underbrace{\left(1-\frac{r^2}{R^2}\right)r\,dr}_{(1-x^2)Rx(R\,dx)}$

$= \boxed{\underbrace{\dfrac{PR^2}{2\eta\ell}}_{2V_{max}}\int_0^1 (1-x^2)\,x\,dx}$ $\quad V_{max} = v(0) = \dfrac{PR^2}{4\eta\ell}$

$\left[\begin{array}{l} x = r/R \\ r = Rx,\ dr = R\,dx \\ r=0 \to x=0 \\ r=R \to x = R/R = 1 \end{array}\right]$

(b) $= 2V_{max}\int_0^1 x - x^3\,dx = 2V_{max}\left(\dfrac{x^2}{2} - \dfrac{x^4}{4}\right)\Big|_0^1 = 2V_{max}\left(\dfrac12 - \dfrac14\right) = \boxed{\tfrac12 V_{max}}$

(c) $\dfrac{V_{max}}{2} = \underbrace{\dfrac{P}{4\eta\ell}}_{=V_{max}/R^2}(R^2-r^2) = V_{max}\dfrac{R^2-r^2}{R^2} \to \dfrac12 = \dfrac{R^2-r^2}{R^2} \to R^2-r^2 = \tfrac12 R^2 \to r^2 = \tfrac12 R^2 \to \boxed{r = \tfrac{1}{\sqrt2}R}$ (≈ 0.707)

d) Maple $\langle v\rangle = \int_0^R \dfrac{P}{8\eta\ell}(R^2-r^2)\,r\,dr = \dfrac{PR^2}{8\eta\ell} = \tfrac12 V_{max}$! yes!

MAT1505-OV02 21F Quiz 2 Remarks

The starting point is $\langle v \rangle = \frac{1}{\pi R^2} \int_0^R \frac{P}{4\eta\ell}(R^2 - r^2)\, 2\pi r\, dr$

$$\underset{\text{Maple}}{=} \frac{PR^2}{\cancel{8}\eta\ell}$$

you could have used Maple before doing anything to see the final result!

OR pull out the constants first:

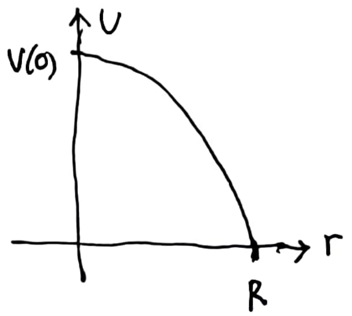
$$\langle v \rangle = \frac{P(2\pi)}{(4\eta\ell)(\pi R^2)} \underbrace{\int_0^R (R^2 - r^2) r\, dr}$$

This is a simple indefinite integral nearly the same as the semicircle area example in the lecture notes:

$$2\int_0^a \sqrt{a^2 - r^2}\, dr$$

(the same in its structure)

Finally notice that the result is just $\langle v \rangle = \frac{1}{2} v(0)$ which is the maximum value of the velocity distribution.



This gives meaning to the final formula which otherwise is just a jumble of letters.

We need to go beyond just getting formulas when we use calc in practice, we need to see how we can interpret them if possible.

AND just because I dont tell you to check with Maple explicitly in a problem, it is a good idea to do so whenever you can.
By the time I asked you to check, you could have already done some manipulation errors.