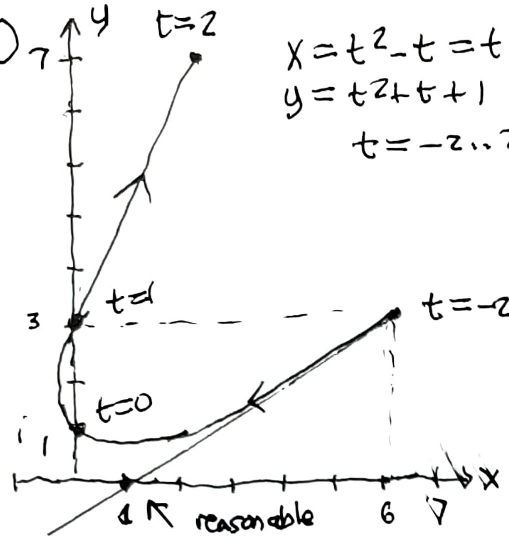
MATV505-01/02 21F Final Exam Answers (1)

(1)



$x = t^2 - t = t(t-1)$
$y = t^2 + t + 1$
$t = -2..2$

a) $t=-2$: $x = -2(-2-1) = 6$
$y = (-2)^2 - 2 + 1 = 3$

$t = 2$: $x = 2(2-1) = 2$
$y = 2^2 + 2 + 1 = 7$

$0 = x = t(t-1) \to t = 0 \to y = 0+0+1 = 1$
$t = 1 \to y = 1+1+1 = 3$

$\boxed{\text{y-intercepts are at 1 and 3}} \to t = 0..1$ parametrizes the curve in the second quadrant

b) $x' = 2t - 1$
$y' = 2t + 1$

$$\frac{dy}{dx} = \frac{y'}{x'} = \frac{2t+1}{2t-1}$$

$$\left.\frac{dy}{dx}\right|_{t=-2} = \frac{2(-2)+1}{2(-2)-1} = \frac{-3}{-5} = \frac{3}{5} = m$$

$(x, y)|_{t=-2} = (6, 3)$ from above.

$y - y_0 = m(x - x_0)$

$y - 3 = \frac{3}{5}(x - 6) = \frac{3}{5}x - \frac{18}{5}$

$y = \underbrace{3 - \frac{18}{5}}_{-\frac{3}{5}} + \frac{3}{5}x = \frac{3x - 3}{5}$

$\boxed{y = \frac{3(x-1)}{5}}$ c) $= 0 \to \boxed{x = 1}$ ← x-intercept of tan line.

b) write down the eqn of the tangent line at $t = -2$

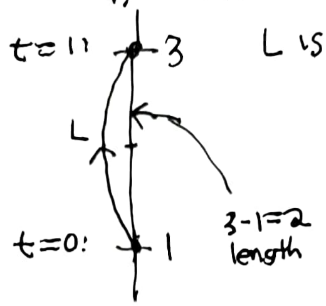
c) Where does this line intersect the horizontal axis?

the curve obviously never intersects the x-axis! what sense would it make to ask if that "curve" (not "line") intersects the x-axis

d) $x'^2 + y'^2 = (2t-1)^2 + (2t+1)^2 =$
$= 4t^2 - \cancel{4t} + 1 + 4t^2 + \cancel{4t} + 1$
$= 8t^2 + 2$

$L = \int_0^1 \sqrt{8t^2+2}\,dt = \boxed{\frac{\sqrt{10}}{2} + \frac{\sqrt{2}}{4} \operatorname{arcsinh}(2) \approx 2.0915}$

d) continued



L is about 10% larger than the secant line between the endpoints, very reasonable

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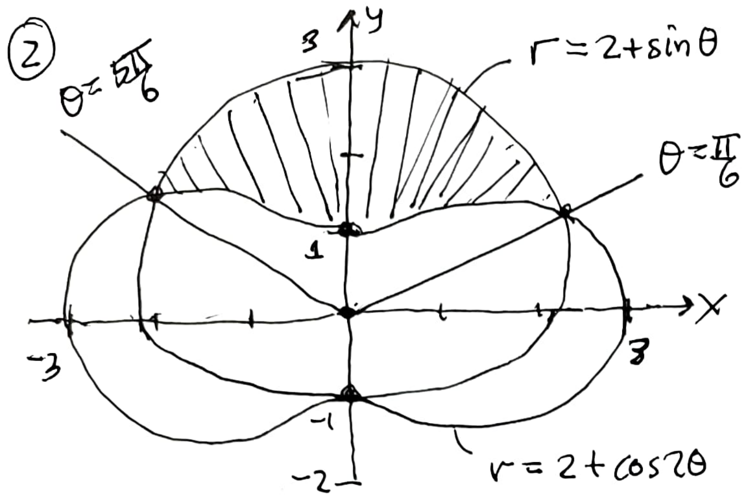
(2)



a) $\cancel{2}+\sin\theta = 2+\cos 2\theta$

$= \cancel{2} + (1-2\sin^2\theta)$

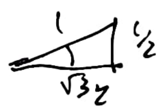
$2\sin^2\theta + \sin\theta - 1 = 0$

$\sin\theta = \dfrac{-1\pm\sqrt{1-4(2)(-1)}}{2(2)} = \dfrac{-1\pm 3}{4}$

$= -1, \frac{1}{2}$

$\theta_0 = \arcsin(-1) = -\frac{\pi}{2}$

$\theta_0 = \arcsin(\frac{1}{2}) = \frac{\pi}{6}$



By symmetry about y-axis

$\boxed{\theta = \frac{\pi}{6}, \frac{5\pi}{6}}$ are intersection pts in upper half plane

$\theta=\frac{\pi}{6} \to r = 2+\sin\frac{\pi}{6} = 2+\frac{1}{2} = \frac{5}{2}$

$x = r\cos\theta = \frac{5}{2}\cos\frac{\pi}{6} = \frac{5}{2}\frac{\sqrt{3}}{2} = \frac{5\sqrt{3}}{4} \approx 2.165$

$y = r\sin\theta = \frac{5}{2}\sin\frac{\pi}{6} = \frac{5}{2}(\frac{1}{2}) = 5/4 \leftarrow$ checks out with above plot (better on original diagram)

$\theta=\frac{5\pi}{6}$: $x = -\frac{5\sqrt{3}}{4}$ so

$$\boxed{\begin{aligned} (r,\theta) &= \left(\tfrac{5}{2}, \tfrac{\pi}{6}\right) \;\&\; (x,y) = \left(\tfrac{5\sqrt{3}}{4}, \tfrac{5}{4}\right) \\ (r,\theta) &= \left(\tfrac{5}{2}, \tfrac{5\pi}{6}\right) \;\&\; (x,y) = \left(-\tfrac{5\sqrt{3}}{4}, \tfrac{5}{4}\right) \end{aligned}}$$

b) $A = \displaystyle\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{(2+\sin\theta)^2}{2} - \frac{(2+\cos 2\theta)^2}{2}\, d\theta$

$= \displaystyle\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}(\cancel{4}+4\sin\theta+\sin^2\theta-\cancel{4}-4\cos 2\theta-\cos^2 2\theta)$

$= \boxed{\displaystyle\int_{\pi/6}^{5\pi/6} \frac{1}{2}(4\sin\theta+\sin^2\theta-4\cos 2\theta-\cos^2 2\theta)\, d\theta}$

continued.

$\boxed{\underset{\text{Maple}}{A =} \frac{51\sqrt{3}}{16} \approx 5.5209}$

d) $A_{circle} = \pi(2)^2 = 4\pi$

$\frac{A}{A_{circle}} \approx 0.44 \lesssim \frac{1}{2}$

looks about right

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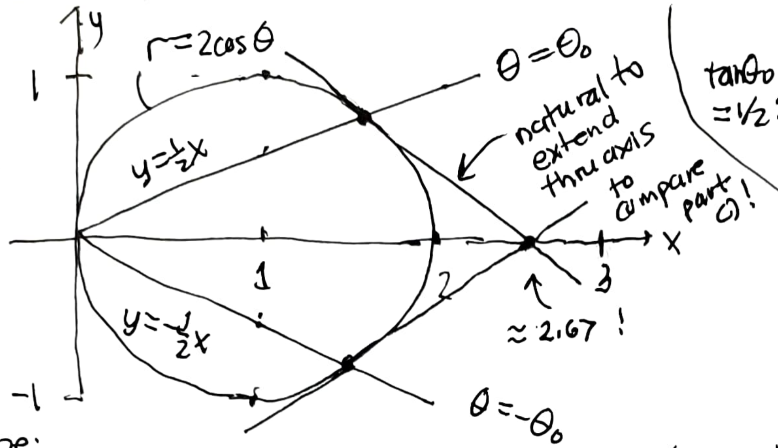
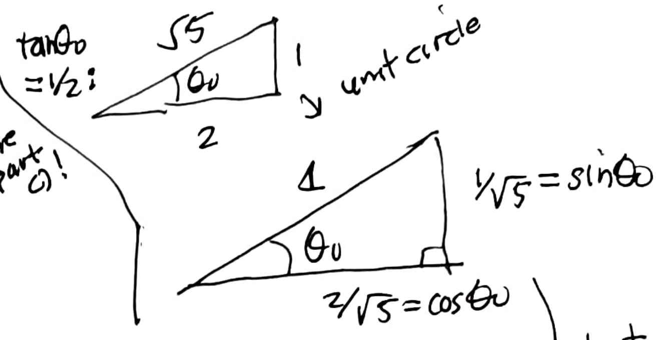
③

a)





but okay to use Maple to evaluate $\cos\theta_0, \sin\theta_0$

b) slope:

$$m = \frac{1}{2} = \tan\theta_0 \to \theta_0 = \arctan\left(\tfrac{1}{2}\right) \to r = 2\cos\left(\arctan\tfrac{1}{2}\right) = 2\left(\frac{2}{\sqrt5}\right) = \frac{4}{\sqrt5}$$

$$x = r\cos\theta_0 = \left(\frac{4}{\sqrt5}\right)\left(\frac{2}{\sqrt5}\right) = \frac{8}{5}$$

$$y = r\sin\theta_0 = \left(\frac{4}{\sqrt5}\right)\left(\frac{1}{\sqrt5}\right) = \frac{4}{5} \to -\frac{4}{5}$$

$\theta = -\theta_0$ by symmetry

$$\boxed{\begin{aligned}(r,\theta) &= \left(\frac{4}{\sqrt5}, \pm\arctan\left(\tfrac{1}{2}\right)\right)\\ (x,y) &= \left(\frac{8}{5}, \pm\frac{4}{5}\right)\end{aligned}}$$

c)

$$x(\theta) = (2\cos\theta)\cos\theta \to x'(\theta) = 4\cos\theta(-\sin\theta) = -4\sin\theta\cos\theta \quad (= -2\sin 2\theta)$$

$$y(\theta) = (2\cos\theta)\sin\theta \to y'(\theta) = 2(\cos^2\theta - \sin^2\theta) \quad (= 2\cos 2\theta)$$

$$\frac{dy}{dx} = \frac{y'}{x'} = \frac{2(\cos^2\theta - \sin^2\theta)}{-4\sin\theta\cos\theta} = -\frac{1}{2}\left(\frac{\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta}\right) \quad \left(= -\cot 2\theta \text{ ! double angle formulas!}\right)$$

$$\left.\frac{dy}{dx}\right|_{\theta=\theta_0} = -\frac{1}{2}\left(\frac{\frac{4}{5} - \frac{1}{5}}{\left(\frac{2}{\sqrt5}\right)\left(\frac{1}{\sqrt5}\right)}\right) = -\frac{1}{2}\left(\frac{3/5}{2/5}\right) = -\frac{3}{4} = m \to +\frac{3}{4} \text{ for lower pt.}$$

$$\theta = \theta_0: y = y_0 + m(x - x_0) = \frac{4}{5} - \frac{3}{4}\left(x - \frac{8}{5}\right) = \underbrace{\frac{4}{5} - \frac{3x}{4} + \cancel{\frac{24}{20}}^{\frac{6}{5}}} = 2 - \frac{3x}{4}$$

$$\theta = -\theta_0: y = -\frac{4}{5} + \frac{3}{4}\left(x - \frac{8}{5}\right) = \ldots \xrightarrow[\text{opp overall sign}]{\frac{4+6}{5} = 2} = -2 + \frac{3x}{4}$$

$$\boxed{y = \pm\left(2 - \frac{3x}{4}\right)} = 0$$

$\to \boxed{x = \frac{8}{3}}$ x-intercept of both tan lines

≈ 2.67 reasonable compared to above plot (natural to extend tangent lines through axis to compare!)

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3 d) $A = \int_{-\theta_0}^{\theta_0} \frac{1}{2}(2\cos\theta)^2 d\theta$

$= \int_{-\theta_0}^{\theta} 2\cos^2\theta \, d\theta \xrightarrow[\text{Maple}]{\theta_0 = \arctan(1/2)} \boxed{\frac{4}{5} + 2\arctan\left(\frac{1}{2}\right)}$

$\boxed{\approx 1.7273}$

$A_{circle} = \pi(1)^2 = \pi$

$\boxed{\frac{A}{A_{circle}} \approx 0.55}$ ~ half and a bit more.

looks like more than ½ so yes!

e) $r = 2\cos\theta$

$r' = -2\sin\theta$

$r^2 + r'^2 = 4\cos^2\theta + 4\sin^2\theta = 4(\cos^2\theta + \sin^2\theta) = 4$

$L = \int_{-\theta_0}^{\theta_0} \sqrt{r^2 + r'^2}\, d\theta = \int_{-\theta_0}^{\theta_0} 2\, d\theta = 2\theta \Big|_{-\theta_0}^{\theta_0} = 4\theta_0 \boxed{= 4\arctan\frac{1}{2} \approx 1.8546}$

$L_{circle} = 2\pi(1) = 2\pi$

$\frac{L}{L_{circle}} \approx 0.30$ about 30% looks good!