MAT1505-01/02 21F Takehome Test 3 Answers (1)

① $f(x) = \sum_{n=0}^{\infty} \underbrace{\frac{(-1)^n (x-3)^n}{(n+1)\,5^n}}_{a_n}$

a) $\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-3|^{n+1}}{(n+2)\,5^{n+1}} \cdot \frac{(n+1)5^n}{|x-3|^n} = \frac{|x-3|}{5}\left(\frac{n+1}{n+2}\right) \xrightarrow{n\to\infty} \frac{|x-3|}{5} < 1$ for convergence

$\boxed{|x-3| < 5 = R} \to x-3 = \pm 5 \to x = -2, 8 \quad \boxed{-2 < x < 8}$

next check endpoints where $|x-3| = 5$, $(x-3) = \pm 5$:

$f(x) \quad \sum_{n=0}^{\infty} \frac{(-1)^n (\pm 5)^n}{(n+1)5^n} = \sum_{n=0}^{\infty} \frac{(-1)^n (\pm 1)^n}{(n+1)}$

$\overset{+1}{=} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \Rightarrow$ convergent alternating harmonic series $(x=8)$

$\overset{-1}{=} \sum_{n=0}^{\infty} \frac{(-1)^{2n}}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1} \Rightarrow$ divergent harmonic series $(x=-2)$

so full interval of convergence: $\boxed{-2 < x \le 8 \text{ or } (-2, 8]}$

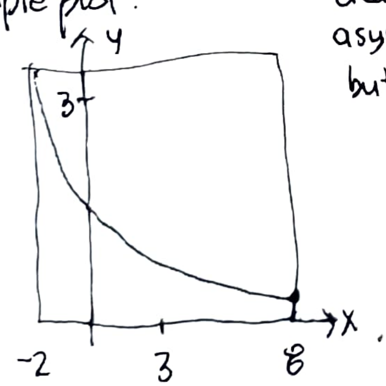
b) $f(4) = \sum_{n=0}^{\infty} \frac{(-1)^n (4-3)^n}{(n+1)5^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)5^n} \underset{\text{Maple}}{=} \boxed{5\ln\left(\frac{6}{5}\right)} \approx 0.9116077840$

$\approx \boxed{0.91161}$

c) assume ($x > 2$ and $x \le 8$):

$f(x) \underset{\text{Maple}}{=} \frac{5\ln\left(\frac{x+2}{5}\right)}{x-3} \quad \left[= \frac{5}{x-3} \ln\left(1 + \frac{x-3}{5}\right) \right.$ algebra tricks from $\ln(1+x) = x + \cdots$, $\left. \frac{\ln(1+x)}{x} = 1 + \cdots \right]$

Maple plot:



clearly vertical asymptote at $x=-2$, but continuous at $x=8$.

agrees with interval of convergence properties.

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(2) $\lim_{x\to 0} \dfrac{\cos(x^2) - e^{x^4}}{\sin(x^4)}$:

$$= \frac{\left(\cancel{1} - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \frac{(x^2)^8}{6!} + \dots\right) - \left(\cancel{1} + (x^4) + \frac{(x^4)^2}{2!} + \dots\right)}{(x^4) - \frac{(x^4)^3}{3!} + \dots}$$

$$= \frac{\left(-\frac{1}{2} - 1\right)x^4 + \left(\frac{1}{4!} - \frac{1}{2!}\right)x^8 + \dots}{x^4 - \frac{x^{12}}{3!} + \dots} \qquad \left(\frac{1}{4\cdot 3} - 1\right)\frac{1}{2} = -\frac{11}{24}$$

$$= \frac{x^4\left(-\frac{3}{2} + -\frac{11}{24}x^4 + \dots\right)}{x^4\left(1 - \frac{x^8}{3!} + \dots\right)} = \frac{-\frac{3}{2} - \frac{11}{24}x^4 + \dots}{1 - \frac{x^8}{6} + \dots} \to \boxed{-\frac{3}{2}} \text{ as } x \to 0$$

Maple agrees!

The Taylor senes for e^x, $\cos x$, $\sin x$ are given!

(3) a) $f(x) = x^{1/2}$ $\qquad f(4) = 4^{1/2} = 2$

$f'(x) = \frac{1}{2}x^{-1/2}$ $\qquad f'(4) = \frac{1}{2}\left(\frac{1}{2}\right) = 1/4$

$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)x^{-3/2}$ $\qquad f''(4) = -\frac{1}{4}\left(\frac{1}{8}\right) = -\frac{1}{32} \to \frac{f''(4)}{2!} = -\frac{1}{64}$

$f'''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^{-5/2}$ $\qquad f'''(4) = \frac{3}{8}\left(\frac{1}{32}\right) = \frac{3}{256} \to \frac{f'''(4)}{3!} = \frac{1}{512}$

$T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(4)}{n!}(x-4)^n$ $\qquad$ oops! $\frac{1}{512}$

$= \boxed{2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3}$

b) $T_3(5) \approx 2.236328125 \approx \boxed{2.23633}$

Maple

c) alternating series estimate: next term

$f^{(4)}(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)x^{-7/2}$

$f^{(4)}(4) = -\frac{15}{2^4}\left(\frac{1}{2^7}\right)$

$\frac{f^{(4)}(4)}{4!} = -\frac{3\cdot 5}{(4\cdot 3\cdot 2)2^{11}} = \frac{-5}{2^{14}} = \frac{-5}{16384} \approx \boxed{-0.0000305}$ error estimate

$\underbrace{\sqrt{5}}_{2.236068} - T_3(5) \underset{\text{Maple}}{\approx} \boxed{-0.000260}$ error true

a bit less in abs value than the estimate as expected.

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(4) a) $f(x) = \sum_{n=0}^{\infty} \left(\frac{(2n)!}{2^{2n}(n!)^2} \right)^2 x^{2n} = \sum_{n=0}^{\infty} a_n$

$$= \left(\frac{0!}{2^0(0!)^2}\right)^2 + \left(\frac{2!}{2^2(1!)^2}\right)^2 x^2 + \left(\frac{4!}{2^4(2!)^2}\right)^2 x^4 + \dots$$

$\frac{4\cdot 3\cdot 2}{2^4 \cdot 2^2} = \frac{3}{8}$

$$= \boxed{1 + \frac{1}{4}x^2 + \frac{9}{64}x^4 + \dots}$$

b) $f(0.5) \underset{\text{Maple}}{\approx} 1.\underbrace{0712}89062 \approx \boxed{1.0713}$

notice that this means the correct period is about 7.1% larger

c) $\left|\frac{a_{n+1}}{a_n}\right| = \left(\frac{(2(n+1))!}{2^{2(n+1)}((n+1)!)^2}\right)^2 |x|^{2(n+1)} \cdot \left(\frac{2^{2n}(n!)^2}{(2n)!}\right)^2 \frac{1}{|x|^{2n}}$

$$= \left(\frac{(2n+2)!}{(2n)!} \cdot \frac{1}{2^2} \cdot \frac{(n!)^2}{((n+1)!)^2}\right)^2 |x|^2$$

$$= \left(\frac{(2n+2)(2n+1)\cancel{(2n)!}}{\cancel{(2n)!}} \frac{(n!)^2}{((n+1)\,n!)^2}\right)^2 \frac{|x|^2}{(2^2)^2}$$

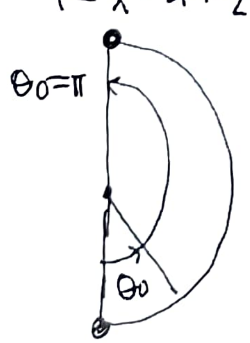
$$= \left(\frac{(2n+2)(2n+1)}{(n+1)^2}\right)^2 \frac{|x|^2}{(2^2)^2}$$

$$= 2^4 \left(\frac{(1+\frac{1}{n})(1+\frac{1}{2n})}{(1+1/n)^2}\right)^2 \frac{|x|^2}{2^4} = \underbrace{\left(\frac{(1+\frac{1}{n})(1+\frac{1}{2n})}{(1+\frac{1}{n})^2}\right)^2}_{\to 1 \text{ as } n\to\infty} |x|^2 \xrightarrow[n\to\infty]{\text{as}} |x|^2 < 1$$

$$\boxed{|x| < 1 = R}$$

so $0 \le x < 1$

$1 = x = \sin\frac{\theta_0}{2} \to \frac{\theta_0}{2} = \frac{\pi}{2} \to \theta_0 = \pi$



This is the $\boxed{\text{upward vertical direction.}}$

If the pendulum swings past this it keeps on rotating forever and there is no period! (at which θ reaches a maximum value).

In fact this is an unstable equilibrium and Maple says $\lim_{x\to 1^-} f(x) = \infty$

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(4) continued.

d) $f(x) = \frac{2\,\text{EllipticK}(x)}{\pi}$ (Maple)

e) $f(0.5) \approx 1.073182007 \approx \boxed{1.0732}$

about 7.3% larger than the high school formula
compared to 7.1% for the 3 term Taylor polynomial value.

f) $n=4$: $T_8(0.5) \approx 1.07311 \approx 1.0731$ falls short

$n=5$: $T_{10}(0.5) \approx 1.07317 \approx 1.0732$ nails it.

$\boxed{2n=10:}$ The 10th degree polynomial approximation is needed to get the 4th decimal place correct