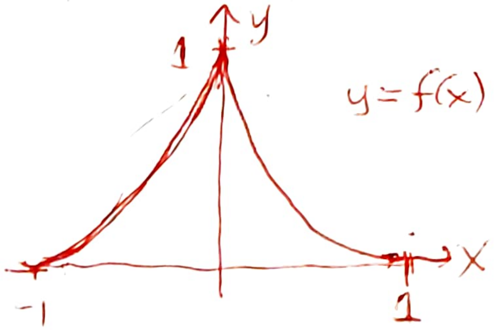
MAT1505-01/02 21F Test 2 Answers

① a) $f(x) = (1-x^{2/3})^{3/2}, \quad -1 \le x \le 1$

$f'(x) = \frac{3}{2}(1-x^{2/3})^{1/2}(0 - \tfrac{2}{3}x^{-1/3})$

$= -\frac{(1-x^{2/3})^{1/2}}{x^{1/3}}$

$1+f'(x)^2 = 1 + \frac{1-x^{2/3}}{x^{2/3}} = \cancel{1} + \underbrace{x^{-2/3} - \cancel{1}}_{= x^{-2/3}}$



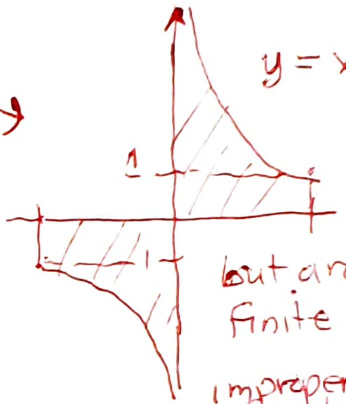
b) $L = \int_0^1 \sqrt{1+f'(x)^2}\,dx = \int_0^1 (x^{-2/3})^{1/2}\,dx$

$= \underbrace{\int_0^1 x^{-1/3}\,dx}_{} = \frac{x^{2/3}}{2/3}\Big|_0^1 = \frac{3}{2} - 0 = \boxed{\frac{3}{2}}$

∞-slope cusp at $x=0$

improper integral! $= \lim_{a\to 0^+} \int_a^1 x^{-1/3}\,dx = \lim_{a\to 0^+} \frac{x^{2/3}}{2/3}\Big|_a^1$

$\left(x^{-1/3} \to \infty \text{ as } x \to 0^+\right) \quad = \lim_{a\to 0^+} \frac{3}{2}(1 - a^{2/3}) = \boxed{\frac{3}{2}}$



but area finite = 2L

improper because of vertical asymptote

c) $s(x) = \int_0^x t^{-1/3}\,dt$

$= \lim_{a\to 0^+} \int_a^x t^{-1/3}\,dt = \lim_{a\to 0^+} \frac{t^{2/3}}{2/3}\Big|_a^x$

$= \lim_{a\to 0^+} \frac{3}{2}(x^{2/3} - a^{2/3}) = \boxed{\frac{3}{2}x^{2/3}}$

d) $s(1) = \frac{3}{2} = L$ ✓

← look for obvious composition in integrand $\frac{1}{(1+e^{3-x})^2} = \frac{1}{u^2}$

② $p(x) = \frac{e^{3-x}}{(1+e^{3-x})^2}$

$P(2 \le x \le 4) = \int_2^4 \frac{\overbrace{e^{3-x}\,dx}^{-du}}{\underbrace{(1+e^{3-x})^2}_{u^2}} = \frac{1}{1+e^{3-x}}\Big|_2^4 = \frac{1}{1+e^{-1}} - \frac{1}{1+e^{1}}$

$= \frac{e}{e+1} - \frac{1}{1+e} = \boxed{\frac{e-1}{e+1}}$ simplest!

$= \frac{1+e^1 - (1+e^{-1})}{(1+e^{-1})(1+e)} = \boxed{\frac{e - e^{-1}}{(1+e^{-1})(1+e)} \approx 0.462}$ yikes!

at least combine to single fraction

$u = 1 + e^{3-x}$

$du = e^{3-x}(-1)\,dx$

$e^{3-x} = u - 1, \quad e^{3-x}dx = -du$

$\int -u^{-2}\,du = -\frac{u^{-1}}{-1} + C$

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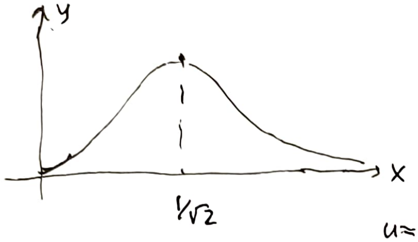
(3) $p(x) = 2xe^{-x^2}, \; x \geq 0$

Note: $0 = p'(x) = 2e^{-x^2} + 2x\, e^{-x^2}(-2x)$

$= 2e^{-x^2}(\underbrace{1-2x^2}_{=0}) \to 2x^2 = 1$

$x^2 = 1/2$

$x = 1/\sqrt{2} \equiv x_{max}$



a) $\int_0^\infty 2xe^{-x^2}dx = \int_{x=0 \to u=0}^{x=\infty \to u=-\infty} \underbrace{e^{-x^2}}_{e^u} \underbrace{(2x\,dx)}_{-du}$

$u = -x^2 \quad < \begin{matrix} x=0 \to u=0 \\ x\to\infty \to u \to -\infty \end{matrix}$

$du = -2x\,dx$

$-\frac{1}{2}du = x\,dx$

$= \int_0^{-\infty} e^u(-du) = \underbrace{\int_{-\infty}^0 e^u du}_{\text{improper integral}} = \lim_{t\to-\infty} \int_t^0 e^u du$

$= \lim_{t\to\infty} e^u \Big|_t^0$

$= \lim_{t\to-\infty} (1-\cancel{e^t}_0) = 1$ ✓

b) $\mu = \int_0^\infty x(2xe^{-x^2})dx$

$= \int_0^\infty 2x^2 e^{-x^2}dx \underset{\text{Maple}}{=} \boxed{\frac{\sqrt{\pi}}{2} \approx 0.886}$

no simple antiderivative formula!

c) $\frac{1}{2} = \int_0^m 2xe^{-x^2}dx = -e^{-x^2}\Big|_0^m = -e^{-m^2} + 1$

$1 - \frac{1}{2} = e^{-m^2}$

$\frac{1}{2} = e^{-m^2}$

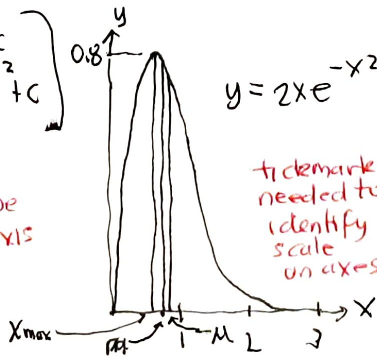
$2 = e^{m^2}$

$\ln 2 = m^2$

$\boxed{m = \sqrt{\ln 2} \approx 0.833} < \mu$

$= \sqrt{-\ln 1/2}$

$\left[\int e^u(-du) = -e^u + c = -e^{-x^2} + c\right]$



x-values should be marked on x-axis

tickmarks needed to identify scale on axes