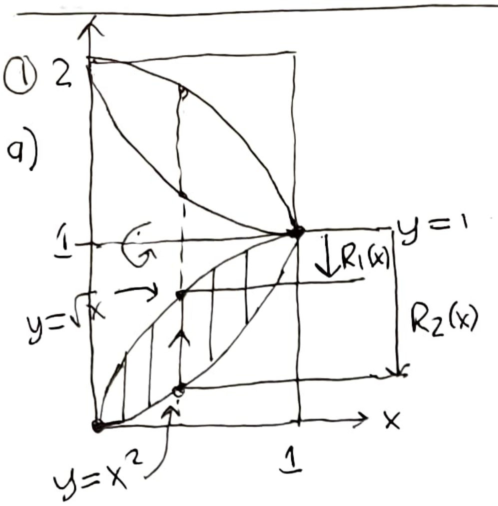
MAT1505-01/02 21F Test 1 Answer key

① a)



$A(x)=\pi(R_2(x)^2-R_1(x)^2)$

$R_1(x)=1-\sqrt{x}$

$R_2(x)=1-x^2$

b) $V=\int_0^1 A(x)\,dx$

$=\int_0^1 \pi\left((1-x^2)^2-(1-\sqrt{x})^2\right)dx$

$=\int_0^1 \pi\left((\cancel{1}-2x^2+x^4)-(\cancel{1}-2\sqrt{x}+x)\right)dx$

$=\boxed{\int_0^1 \pi(-2x^2+x^4+2x^{1/2}-x)\,dx}$

c) $=\pi\left(-\frac{2x^3}{3}+\frac{x^5}{5}+\frac{2x^{3/2}}{3/2}-\frac{x^2}{2}\right)\Big|_0^1=\pi\underbrace{\left(-\frac{2}{3}+\frac{1}{5}+\frac{4}{3}-\frac{1}{2}\right)}_{\text{dangerous, check with Maple!}}=\pi\left(\frac{1}{5}+\frac{1}{6}\right)=\boxed{\frac{11}{30}\pi}$

d) Maple agrees with this starting from either integral

② a) $\frac{dW}{dt}=200-4t=0\to t=\frac{200}{4}=50$ must stop here or no longer flowing out!

or: $r(50)=200-4(50)=0$ $\boxed{\text{the water stops flowing out}}$

b) $\Delta W=\int_0^{10} r(t)\,dt=\int_0^{10} 200-4t\,dt=200t-2t^2\Big|_0^{10}=200(10)-2(100)$
$=2000-200$

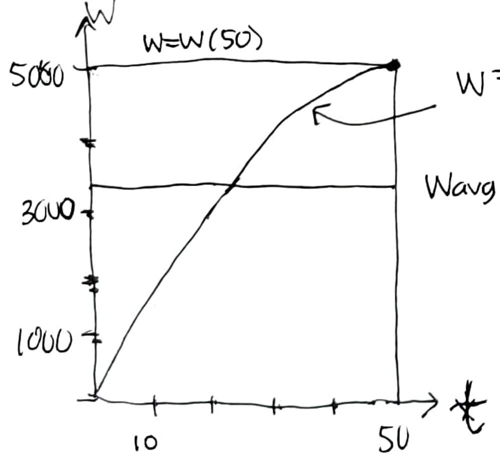
$=\boxed{1800 \text{ liters}}$

c) $W(t)=\int_0^t (200-4u)\,du=200u-2u^2\Big|_0^t=\boxed{200t-2t^2}$ (liters)

$W(50)=200(50)-2(50)^2=\boxed{5000 \text{ liters}}$

d) $W_{avg}=\frac{1}{50}\int_0^{50} W(t)\,dt=\frac{1}{50}\int_0^{50} 200t-2t^2\,dt=\frac{1}{50}\left(200t^2-\frac{2t^3}{3}\right)\Big|_0^{50}$

$\underset{\text{Maple}}{=}\frac{10000}{3}\boxed{\begin{aligned}&=3333\tfrac{1}{3}\text{ liters}\\&\approx 3333\text{ liters}\end{aligned}}$



W=W(t) parabola, vertex at t=50 so slope vanishes
(r(t)=0)

looks great!
area below seems to balance area above W=Wavg

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③ a) $A_{peak} = \int_0^a \underbrace{x}_{au} e^{\overbrace{-\frac{x^2}{2a^2}}^{-\frac{u^2}{2}}} \underbrace{dx}_{a\,du} = \int_0^1 au\, e^{-\frac{u^2}{2}} a\,du = \boxed{a^2 \int_0^1 u e^{-u^2/2} du}$

$u = \frac{x}{a} \to x = au$

$du = \frac{dx}{a} \to dx = a\,du$

$x = 0 \to u = 0$

$x = a \to u = \frac{a}{a} = 1$

$\int_0^1 u\, e^{\underbrace{-u^2/2}_{w}} \overbrace{u\,du}^{-dw} = \int_0^{-1/2} e^w(-dw) = \int_{-1/2}^0 e^w dw = e^w \Big|_{-1/2}^0$

$w = -\frac{u^2}{2}$

$dw = -\frac{1}{2} 2u\,du = -u\,du$

$u = 0 \to w = 0$

$u = 1 \to w = -\frac{1}{2}$

$= \boxed{1 - e^{-1/2}}$

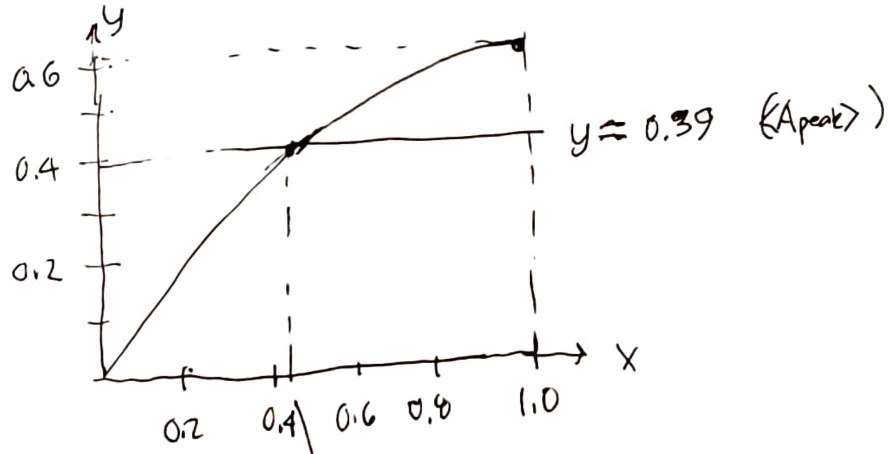
so $\boxed{A_{peak} = a^2(1 - e^{-1/2})}$

$\approx 0.3935\, a^2$

$\langle A_{peak} \rangle = \frac{1}{a} \int_0^a x e^{-\frac{x^2}{2a^2}} dx = \frac{1}{a} a^2 (1 - e^{-1/2}) = \boxed{a(1 - e^{-1/2})}$

$\approx 0.3935\, a$

d)



optional

e) $x \approx 0.43194$

solve on $[0, 1]$:

$x e^{-\frac{x^2}{2}} = 1 - e^{-1/2}$