MAT 2500-01/02 20S Final Exam Answers (1)

① $\int_C (1+x^2y)\,dx + xy\,dy$

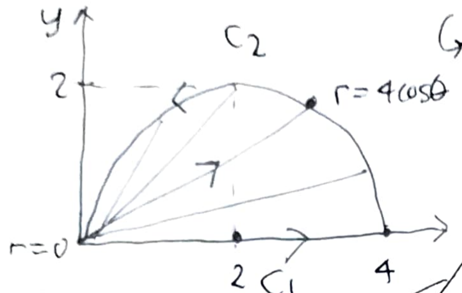
$\to \vec{F} = \langle 1+x^2y,\ xy\rangle$, $x^2+y^2 = 4x$

$r^2 = 4(r\cos\theta)$

$\hookrightarrow r = 4\cos\theta$

$\theta = 0..\pi/2$

correct orientation



$C_1: \vec{r}(t) = \langle t, 0\rangle$, $t = 0..4$

$\vec{r}'(t) = \langle 1, 0\rangle$

$\vec{F}(\vec{r}(t)) = \langle 1, 0\rangle$

$\vec{F}(\vec{r}(t))\cdot\vec{r}'(t) = \langle 1,0\rangle\cdot\langle 1,0\rangle = 1$

$\int_0^4 \vec{F}(\vec{r}(t))\cdot\vec{r}'(t)\,dt = \int_0^4 1\,dt = t\Big|_0^4 = \boxed{4} = \int_{C_1}\vec{F}\cdot d\vec{r}$

C_2:

a) $\vec{r} = \langle x, y\rangle = \langle r\cos\theta, r\sin\theta\rangle$

$= \langle 4\cos^2\theta,\ 4\cos\theta\sin\theta\rangle$

$\vec{r}\,' = \langle -8\cos\theta\sin\theta,\ 4(-\sin^2\theta+\cos^2\theta)\rangle$

$= 4\langle -2\sin\theta\cos\theta,\ \cos^2\theta-\sin^2\theta\rangle$

$\vec{F}(\vec{r}(\theta)) = \langle 1 + (4\cos^2\theta)^2(4\cos\theta\sin\theta),\ 4\cos^2\theta(4\cos\theta\sin\theta)\rangle$

$= \langle 1 + 4^3\cos^5\theta\sin\theta,\ 4^2\cos^3\theta\sin\theta\rangle$

$\vec{F}(\vec{r}(\theta))\cdot\vec{r}\,'(\theta) = 4\big[(-2\sin\theta\cos\theta)(1+4^3\cos^5\theta\sin\theta)$

$+ (\cos^2\theta-\sin^2\theta)(4^2\cos^3\theta\sin\theta)\big]$

$= -8\sin\theta\cos\theta\,(1+64\cos^5\theta\sin\theta - 16\cos^4\theta + 8\cos^2\theta\sin^2\theta)$

Maple's simplify replaces $1-\cos^2\theta$, combines $\cos^4 t$

$= -8\sin t\cos t\,[64\cos^5 t\sin t - 16\cos^4 t + 8\cos^2 t + 1]$

$\int_0^\pi \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt \underset{\text{Maple}}{=} \boxed{\frac{4}{3} - 10\pi} = \int_{C_2}\vec{F}\cdot d\vec{r}$

$\int_C \vec{F}\cdot d\vec{r} = \left(\frac{4}{3} - 10\pi\right) + 4$

$= \boxed{\frac{16}{3} - 10\pi}$ final answer

b) $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$

$= \frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(1+x^2y)$

$= y - x^2$

$= r\sin\theta - (r\cos\theta)^2$

b) $\int_0^{\frac{\pi}{2}}\int_0^{4\cos\theta} (r\sin\theta - r^2\cos^2\theta)\,r\,dr\,d\theta = \int_0^{\frac{\pi}{2}}\int_0^{4\cos\theta}(r^2\sin\theta - r^3\cos^2\theta)\,dr\,d\theta$

$\underset{\text{Maple}}{=} \boxed{\frac{16}{3} - 10\pi}$ ✓ agrees

MAT 2500-01/02 20S Final Exam Answers (2)

$\vec F(x,y,z) = \langle 3x+y,\ x+3y,\ 3z\rangle$

(2) a) $\vec r(t) = \langle t\cos t,\ t\sin t,\ t^2\rangle = \langle x, y, z\rangle$

$\vec r\,'(t) = \langle \cos t - t\sin t,\ \sin t + t\cos t,\ 2t\rangle$

$\vec F(\vec r(t)) = \langle \underbrace{3t\cos t}_{3x} + \underbrace{t\sin t}_{y},\ \underbrace{t\cos t}_{x} + \underbrace{3\sin t}_{3y},\ \underbrace{3t^2}_{3z}\rangle$

$$\vec F(\vec r(t))\cdot\vec r\,'(t) = (3t\cos t + t\sin t)(\cos t - t\sin t) + (t\cos t + 3t\sin t)(\sin t + t\cos t) + (2t)(3t^2)$$

$$= 3t\cos^2 t + t\sin t\cos t - 3t^2\cancel{\cos t\sin t} - t^2\sin^2 t + 6t^3 + 3t\sin^2 t + t\sin t\cos t + 3t^2\cancel{\cos t\sin t} + t^2\cos^2 t$$

$$= 3t + 2t\sin t\cos t + 6t^3 + t^2(\cos^2 t - \underbrace{\sin^2 t}_{1-\cos^2 t})$$ Maple simplify

$= t(3 + 2\sin t\cos t + 6t^2 + 2t\cos^2 t - t)$ maple simplify.

$$\int_0^{2\pi} \vec F(\vec r(t))\cdot\vec r\,'(t)\,dt = \dots = \boxed{6\pi^2 + 24\pi^4} = 6\pi^2(1+4\pi^2)$$

$\vec r(t) = \langle t, 0, t^2\rangle \qquad \vec r\,'(t) = \langle 1, 0, 2t\rangle$

$\vec F(\vec r(t)) = \langle 3t, t, 3t^2\rangle$

$\vec F(\vec r(t))\cdot\vec r\,'(t) = 3t + 6t^3$

$$\int_0^{2\pi} \vec F(\vec r(t))\cdot\vec r\,'(t)\,dt = \int_0^{2\pi} 3t + 6t^3\,dt = \frac{3}{2}t^2 + \frac{6}{4}t^4\Big|_0^{2\pi}$$

$= \boxed{6\pi^2 + 24\pi^4}$ agrees.

b) $\operatorname{curl}\vec F = \langle \underbrace{\frac{\partial}{\partial y}(3z)}_{0} - \underbrace{\frac{\partial}{\partial z}(x+3y)}_{0},\ \underbrace{\frac{\partial}{\partial z}(3x+y)}_{0} - \underbrace{\frac{\partial}{\partial x}(3z)}_{0},\ \frac{\partial}{\partial x}(x+3y) - \frac{\partial}{\partial y}(3x+y)\rangle$ $\quad 1 - 1 = 0$

$= \langle 0,0,0\rangle$ everywhere so $\vec F = \vec\nabla f$

$\int\left[\frac{\partial f}{\partial x} = 3x+y\right]dx \to f = \int 3x+y\,dx = \frac{3}{2}x^2 + xy + C(y,z)$

$\frac{\partial f}{\partial y} = x + 3y \quad == \quad \frac{\partial f}{\partial y} = 0 + \cancel{x} + \frac{\partial C}{\partial y}(y,z) \to \int\left[\frac{\partial C}{\partial y}(y,z) = 3y\right]dy$

$\frac{\partial f}{\partial z} = 3z$

$C(y,z) = \int 3y\,dy = \frac{3y^2}{2} + C(z)$

$f = \frac{3}{2}x^2 + xy + \frac{3}{2}y^2 + C(z)$

$= \quad \frac{\partial f}{\partial z} = 0 + C'(z) \to \int\left[C'(z) = 3z\right]dz$

$C(z) = \frac{3}{2}z^2 + k$

$$\boxed{f = \frac{3}{2}x^2 + xy + \frac{3}{2}y^2 + \frac{3}{2}z^2 + k}$$

MAT2500-01/02 20S Final Exam Answers (3)

2 c) set $k=0$: $f=\frac{1}{2}(3x^2+2xy+3y^2+3z^2)$

For both curves: $\vec{r}(0)=\langle 0,0,0\rangle$, $\vec{r}(2\pi)=\langle 2\pi,0,(2\pi)^2\rangle$

$f(0,0,0)=0$ $\qquad f(2\pi,0,4\pi^2)=\frac{1}{2}(3\cdot 4\pi^2+3(4\pi^2)^2)$

$=\boxed{6\pi^2+24\pi^4}$ agrees

3 a) $\vec{F}(x,y)=\dfrac{k\langle x,y\rangle}{x^2+y^2}$

$|\vec{F}(x,y)|=\dfrac{k}{x^2+y^2}\sqrt{x^2+y^2}=\dfrac{k}{\sqrt{x^2+y^2}}\quad\left(=\dfrac{k}{r}\right)$ inversely proportional to distance from origin, goes to ∞ as $r\to 0$

b) $\hat{k}\cdot(\mathrm{curl}\,\vec{F})=$ $\dfrac{\partial}{\partial x}\left(\dfrac{ky}{x^2+y^2}\right)-\dfrac{\partial}{\partial y}\left(\dfrac{kx}{x^2+y^2}\right)$

$=ky(-1)(x^2+y^2)^{-2}(2x)-kx(-1)(x^2+y^2)^{-2}(2y)$

$=\dfrac{2k}{(x^2+y^2)^2}(-xy+xy)=\boxed{0}$ if $(x,y)\neq(0,0)$
undefined at $(0,0)$.

c) $\mathrm{div}\,\vec{F}=$ $\dfrac{\partial}{\partial x}\left(\dfrac{kx}{x^2+y^2}\right)+\dfrac{\partial}{\partial y}\left(\dfrac{ky}{x^2+y^2}\right)=k\left[\dfrac{(x^2+y^2)\,1-x(2x)}{(x^2+y^2)^2}\right]+k\left[\dfrac{(x^2+y^2)\cdot 1-y(2y)}{(x^2+y^2)^2}\right]$

$=\dfrac{k}{(x^2+y^2)^2}\left[(y^2-x^2)+(x^2-y^2)\right]=\boxed{0}$ if $(x,y)\neq(0,0)$
undefined at $(0,0)$

d) from b) a potential function exists (except at $(0,0)$): $\vec{F}=\vec{\nabla}f$

$\int\left[\dfrac{\partial f}{\partial x}=\dfrac{kx}{x^2+y^2}\right]dx\to f=k\underbrace{\int\dfrac{x\,dx}{x^2+y^2}}_{\frac{1}{2}\int\frac{du}{u}=\frac{1}{2}\ln|u|}=\frac{1}{2}\ln(x^2+y^2)+C(y)$

$\dfrac{\partial f}{\partial y}=\dfrac{ky}{x^2+y^2}$ $\qquad \dfrac{\partial f}{\partial y}=\dfrac{1}{2}\dfrac{k}{x^2+y^2}(2y)+C'(y)=\dfrac{ky}{x^2+y^2}+C'(y)\to C'(y)=0$

$C(y)=k$

$f=\frac{1}{2}\ln(x^2+y^2)+k$
$=\boxed{\ln\sqrt{x^2+y^2}+k}=\ln r+k$

e) $\vec{r}(t)=\langle a\cos t,a\sin t\rangle$, $t=0..2\pi$
$\vec{r}\,'(t)=\langle -a\sin t,a\cos t\rangle$
$\hat{T}(t)=\langle -\sin t,\cos t\rangle$
$\hat{N}(t)=\langle\cos t,\sin t\rangle$ radially outward ✓ $\qquad ds=|\vec{r}\,'(t)|\,dt=a\,dt$

$\vec{F}(\vec{r}(t))\cdot\hat{N}(t)=k\dfrac{\langle a\cos t,a\sin t\rangle}{a^2}\cdot\langle\cos t,\sin t\rangle=\dfrac{ka}{a^2}(\cos^2 t+\sin^2 t)=\dfrac{k}{a}$

$\int_C F_{\parallel}\,ds=\int_0^{2\pi}\left(\dfrac{k}{a}\right)a\,ds=k(2\pi)=\boxed{2\pi k}$ independent of a! ⟶

If Greens thm. applied, it should be zero since $\mathrm{div}\,\vec{F}=0$ except at $(0,0)$.
and not zero!