MAT2705-04/05 20F Quiz 9 Print Name (Last, First) ______________________________|__

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (**but give decimal approximations for interpretation**). Indicate where technology is used and what type (Maple, GC). **Use Maple for the matrix products.**

1. For the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 7 & 2 \\ 10 & -15 & -4 \end{bmatrix}$, a) show all the steps in the diagonalization process, ordering the real eigenvalues by increasing order when distinct, and scaling the eigenvectors by a positive number to get integer components if necessary, ending with evaluating the matrix product which diagonalizes the matrix.

b) Use the eigenbasis to find the new coordinates/components of the point/vector $\vec{x} = \langle 1, -1, 2 \rangle$. Then express this vector as a linear combination of the eigenvectors and finally evaluate the linear combination to show that it reduces to the original vector.

2. For the matrix $A = \begin{bmatrix} -2 & 0 & 2 \\ 2 & -4 & 0 \\ 0 & 4 & -2 \end{bmatrix}$, a) show all the steps in the diagonalization process, putting the single real eigenvalue first, and ordering the complex conjugate eigenvalue pair by positive imaginary part first, negative imaginary part second, scaling up by positive numbers if necessary to get integer entries, ending with evaluating the matrix product which diagonalizes the matrix.

b) Use the eigenbasis to find the new coordinates/components of the point/vector $\vec{x} = \langle 10, -15, 30 \rangle$. Then express this vector as a linear combination of the eigenvectors $\vec{x} = y_1 \vec{b}_1 + y_2 \vec{b}_2 + y_3 \vec{b}_3$ and finally evaluate the linear combination to show that it reduces to the original vector.

c) Re-express the components of the complex vector $y_2 \vec{b}_2$ in polar form $A\, e^{I\theta}$.

d) Now evaluate the real and imaginary parts of the following complex vector function (using the identity $e^{(p+qI)t} = e^{pt} e^{Iqt} = e^{pt}(\cos(qt) + I\sin(qt))$ using those values of the new coordinates that you found in part b).

$\vec{Z} = e^{\lambda_2 t} \vec{b}_2 = \vec{X}_1 + I\vec{X}_2$, and then the real part of $2\, y_2 \vec{Z}$.

① a) $|A-\lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ -4 & 7-\lambda & 2 \\ 10 & -15 & -4-\lambda \end{vmatrix} \underset{\text{Maple}}{=} -\lambda^3+4\lambda^2-5\lambda+2 = -(\lambda-1)^2(\lambda-2) = 0$

$\lambda = 1$, $m=2$; $\lambda = 2$, $m=1$

$\lambda=1$: $A-I = \begin{bmatrix} 1-1 & 0 & 0 \\ -4 & 7-1 & 2 \\ 10 & -15 & -4-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -4 & 6 & 2 \\ 10 & -15 & -5 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 \\ -2 & 3 & 1 \\ 2 & -3 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -3/2 & -1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

L F F

$x_2 = t_1, x_3 = t_2, x_1 = +\frac{3}{2}t_1 + \frac{1}{2}t_2$

$\langle x_1, x_2, x_3 \rangle = \langle \frac{3}{2}t_1 + \frac{1}{2}t_2, t_1, t_2 \rangle$

$= t_1 \langle \frac{3}{2}, 1, 0 \rangle + t_2 \langle \frac{1}{2}, 0, 1 \rangle$

$\vec{b}_1 = \langle 3, 2, 0 \rangle$ $\vec{b}_2 = \langle 1, 0, 2 \rangle$

$\lambda=2$: $A-2I = \begin{bmatrix} 1-2 & 0 & 0 \\ -4 & 7-2 & 2 \\ 10 & -15 & -4-2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ -4 & 5 & 2 \\ 10 & -15 & -6 \end{bmatrix}$

$\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 2 \\ 0 & -15 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2/5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

L L F

$x_3 = t$, $x_1 = 0$, $x_2 = -2/5\, t$ $\quad \langle x_1, x_2, x_3 \rangle = \langle 0, -\frac{2}{5}t, t \rangle = t \langle 0, -\frac{2}{5}, 1 \rangle$

$\vec{b}_3 = \langle 0, -2, 5 \rangle$

MAT2705-04/05 20F Quiz 9 Answers (2)

① b) continued $B = \langle b_1 | b_2 | b_3 \rangle = \begin{bmatrix} 3 & 1 & 0 \\ 2 & 0 & -2 \\ 0 & 2 & 5 \end{bmatrix}$, $A_B = B^{-1}AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$$B^{-1} = \frac{1}{2}\begin{bmatrix} 4 & -5 & -2 \\ -10 & 15 & 6 \\ 4 & -6 & -2 \end{bmatrix}$$

b) $x = \langle 1,-1,2 \rangle \rightarrow y = B^{-1}x = \frac{1}{2}\begin{bmatrix} 4 & -5 & -2 \\ -10 & 15 & 6 \\ 4 & -6 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4+5-4 \\ -10-15+12 \\ 4+6-4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 5 \\ -13 \\ 6 \end{bmatrix} = \begin{bmatrix} 5/2 \\ -13/2 \\ 3 \end{bmatrix}$

$$\boxed{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \frac{5}{2}\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} - \frac{13}{2}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix}} = \frac{1}{2}\begin{bmatrix} 15-13+0 \\ 10-0-12 \\ 0-26+30 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \checkmark$$

② a) $|A-\lambda I| = \begin{vmatrix} -2-\lambda & 0 & 2 \\ 2 & -4-\lambda & 0 \\ 0 & 4 & -2-\lambda \end{vmatrix} = -\lambda^3 - 8\lambda^2 - 20\lambda = -\lambda(\lambda^2+8\lambda+20) = 0$

$\lambda = 0, -4+2i, -4-2i$

$\lambda = 0$: $A = \begin{bmatrix} -2 & 0 & 2 \\ 2 & -4 & 0 \\ 0 & 4 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 1 & -2 & 0 \\ 0 & 2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

L L F

$x_3 = t$, $x_1 = t$, $x_2 = \frac{1}{2}t$

$\langle x_1, x_2, x_3 \rangle = t\langle 1, \frac{1}{2}, 1 \rangle$

$b_1 = \langle 2, 1, 2 \rangle \leftarrow 2\vec{b}_1$

$\lambda = -4+2i$

$A + (4-2i)I = \begin{bmatrix} -2+4-2i & 0 & 2 \\ 2 & -4+4-2i & 0 \\ 0 & 4 & -2+4-2i \end{bmatrix} = \begin{bmatrix} 2-2i & 0 & 2 \\ 2 & 2-2i & 0 \\ 0 & 4 & 2-2i \end{bmatrix} \rightarrow \begin{bmatrix} 1-i & 0 & 1 \\ 1 & 1-i & 0 \\ 0 & 2 & 1-i \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1-i & 0 \\ 0 & 1 & \frac{1}{2}(1-i) \\ 0 & -2+2i & 1 \end{bmatrix}$

$-2+2i = -(1-i)^2$

Maple $\rightarrow$ (too tedious by hand)

$\begin{bmatrix} 1 & 0 & \frac{1}{2}+\frac{i}{2} \\ 0 & 1 & \frac{1}{2}-\frac{i}{2} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\quad x_1 = -\frac{1+i}{2}t$, $\quad x_2 = -\frac{1-i}{2}t$

L L F $\quad x_3 = t$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} -\frac{(1+i)}{2} \\ -\frac{(1-i)}{2} \\ 1 \end{bmatrix}$

$\vec{b}_2 = \begin{bmatrix} -(1+i) \\ -(1-i) \\ 2 \end{bmatrix}$

$$B = \langle b_1 | b_2 | b_3 \rangle = \begin{bmatrix} 2 & -1-i & -1+i \\ 1 & -1+i & -1-i \\ 2 & 2 & 2 \end{bmatrix}$$

$$B^{-1} = \frac{1}{20}\begin{bmatrix} 4 & 4 & 4 \\ -2+4i & -2-6i & 3-i \\ -2-4i & -2+6i & 3+i \end{bmatrix} \qquad A_B = B^{-1}AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -4+2i & 0 \\ 0 & 0 & -4-2i \end{bmatrix}$$

MAT 2705-04/05 Quiz 9 Answers (3)

② b) $\vec{x} = \langle 10,-15,30\rangle = 5\langle 2,-3,6\rangle$

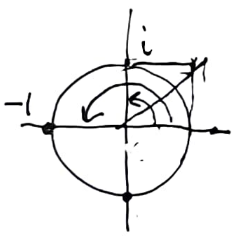
$$y = B^{-1}x = \frac{5}{20}\begin{bmatrix} 4 & 4 & 4 \\ -2+4i & -2-6i & 3-i \\ -2-4i & -2+6i & 3+i \end{bmatrix}\begin{bmatrix} 2 \\ -3 \\ 6\end{bmatrix} = \frac{1}{4}\begin{bmatrix} 8-12+24 \\ (-4+8i)+(6+18i)+(18-6i) \\ (-4-8i)+(6-18i)+(18+6i)\end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 20 \\ 20+20i \\ 20-20i\end{bmatrix} = \begin{bmatrix} 5 \\ 5+5i \\ 5-5i\end{bmatrix}$$

$$\begin{bmatrix} 10 \\ -15 \\ 30\end{bmatrix} = 5\begin{bmatrix} 2 \\ 1 \\ 2\end{bmatrix} + (5+5i)\begin{bmatrix} -1-i \\ -1+i \\ 2\end{bmatrix} + (5-5i)\begin{bmatrix} -1+i \\ -1-i \\ 2\end{bmatrix} = 5\begin{bmatrix} 2 - (1+i)^2 - (1-i)^2 \\ 1 - (1^2-i^2) - (1^2-i^2) \\ 2 + 2(1+i) + 2(1-i)\end{bmatrix}$$

$$= 5\begin{bmatrix} 2 - 2i + 2i \\ 1 - 2 - 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 10 \\ -15 \\ 30\end{bmatrix} \checkmark$$

c) $$y_2 b_2 = 5(1+i)\begin{bmatrix} -1-i \\ -1+i \\ 2\end{bmatrix} = 5\begin{bmatrix} -(1+i)^2 \\ -1^2+i^2 \\ 2+2i\end{bmatrix} = 5\begin{bmatrix} -2i \\ -2 \\ 2+2i\end{bmatrix} = 10\begin{bmatrix} -i \\ -1 \\ 1+i\end{bmatrix} = 10\begin{bmatrix} e^{-i\pi/2} \\ e^{i\pi} \\ \sqrt{2}e^{i\pi/4}\end{bmatrix}$$



$$= \begin{bmatrix} 10e^{-i\pi/2} \\ 10e^{i\pi} \\ 10\sqrt{2}e^{i\pi/4}\end{bmatrix}$$

d) $$Z = e^{-4t}(\cos 2t + i\sin 2t)\begin{bmatrix} -1-i \\ -1+i \\ 2\end{bmatrix} = e^{-4t}\begin{bmatrix} -\cos 2t + \sin 2t + i(-\cos 2t - \sin 2t) \\ -\cos 2t - \sin 2t + i(\cos 2t - \sin 2t) \\ 2\cos 2t + i(2\sin 2t)\end{bmatrix}$$

$$= \underbrace{e^{-4t}\begin{bmatrix} -\cos 2t + \sin 2t \\ -\cos 2t - \sin 2t \\ 2\cos 2t\end{bmatrix}}_{X_1} + i\,\underbrace{e^{-4t}\begin{bmatrix} -\cos 2t - \sin 2t \\ \cos 2t - \sin 2t \\ 2\sin 2t\end{bmatrix}}_{X_2}$$

$$y_2 Z = 5(1+i)(X_1 + iX_2) = 5(X_1 - X_2 + i(X_1 + X_2))$$

$$\begin{bmatrix} (-c+s) - (-c-s) \\ (-c-s) - (c-s) \\ 2c - 2s\end{bmatrix} + i\begin{bmatrix} (-c+s) + (-c-s) \\ (-c-s) + (c-s) \\ 2c + 2s\end{bmatrix} = \begin{bmatrix} 2s \\ -2c \\ 2c-2s\end{bmatrix} + i\begin{bmatrix} -2c \\ -2s \\ 2c-2s\end{bmatrix}$$

$$2\,\mathrm{Re}(y_2 Z) = e^{-4t}\begin{bmatrix} 10\sin 2t \\ -10\cos 2t \\ 10\cos 2t - 10\sin 2t\end{bmatrix}$$

not required: compare with c):

$$\begin{bmatrix} 10\cos(2t - \pi/2) \\ 10\cos(2t + \pi) \\ 10\sqrt{2}\cos(2t + \pi/4)\end{bmatrix}$$