MAT2705-04/05 20F Quiz 6 Print Name (Last, First) ______________________

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC).

1. Express this linear system in matrix form, evaluate the determinant of the coefficient matrix, and use the (technology delivered) inverse matrix to solve the system (state that matrix explicitly):
$x_1 \; -2\,x_3 = -3\,,\; x_1 + x_3 = 9,\; x_2 - 4\,x_4 = 0,\; x_2 + 2\,x_4 = 0\;.$

2. a) Express the vector $\langle 4, 7\rangle$ as a linear combination of the vectors $\{\langle 2, 1\rangle, \langle 1, 3\rangle\}$.
b) Show that this linear combination evaluates to the target vector.

oops! correction

3. a) Express the vector $\langle 1,-8,1\rangle$ as a linear combination of the vectors $\{\langle 1, 2, 1\rangle, \langle 2,-1, 2\rangle\}$.
b) Show that this linear combination evaluates to the target vector.
c) Repeat the problem for a general vector $\langle x_1, x_2, x_3\rangle$ and row reduce the matrix only pivoting on the upper left corner entry by hand (two steps) to obtain the condition that a solution exist in the final entry of the last row (the linear combination of the variables which must vanish to guarantee a solution). Write down this condition, which is the equation of the plane of these two vectors.

▶ solution

①
$$\begin{aligned} x_1 \quad -2x_3 \quad &= 0 \\ x_1 \quad + x_3 \quad &= 0 \\ x_2 \quad -4x_4 &= 0 \\ x_2 \quad +2x_4 &= 0 \end{aligned}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -4 \\ 0 & 1 & 0 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} -3 \\ 9 \\ 0 \\ 0 \end{bmatrix}}_{b}$$

$\det(A) = -18$

$$A^{-1} = \frac{1}{6}\begin{bmatrix} 2 & 4 & 0 & 0 \\ 0 & 0 & 2 & 4 \\ -2 & 2 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$x = A^{-1}b$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 2 & 4 & 0 & 0 \\ 0 & 0 & 2 & 4 \\ -2 & 2 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ 9 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -6+36 \\ 0 \\ 6+18 \\ 0 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 30 \\ 0 \\ 24 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 4 \\ 0 \end{bmatrix}$$

② a) $\langle\langle 2,1\rangle | \langle 1,3\rangle\rangle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix} \to \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix} \to \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{6-1}\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 7 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 12-7 \\ -4+14 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\begin{bmatrix} 4 \\ 7 \end{bmatrix} = 1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 3 \end{bmatrix} \overset{b)}{=} \begin{bmatrix} 2+2 \\ 1+6 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$ ✓

③ a) $\langle\langle 1,2,1\rangle | \langle 2,-1,2\rangle\rangle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -8 \\ 1 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -8 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & -8 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow[\text{Maple}]{\text{rref}} \begin{array}{c} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \\ x_1 \; x_2 \end{array} \to \begin{aligned} x_1 &= -3 \\ x_2 &= 2 \end{aligned} \to \begin{bmatrix} 1 \\ -8 \\ 1 \end{bmatrix} = -3\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \overset{b)}{=} \begin{bmatrix} -3+4 \\ -6-2 \\ -3+4 \end{bmatrix} = \begin{bmatrix} 1 \\ -8 \\ 1 \end{bmatrix}$$

c)
$$\begin{bmatrix} 1 & 2 & x_1 \\ 2 & -1 & x_2 \\ 1 & 2 & x_3 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & x_1 \\ 0 & -5 & x_2 - 2x_1 \\ 0 & 0 & x_3 - x_1 \end{bmatrix}$$

$\to x_3 - x_1 \neq 0 \to$ system inconsistent so we must have $\boxed{x_3 - x_1 = 0}$

both vectors obviously satisfy this condition!

BONUS

$R_2 \to -\frac{1}{5}R_2$

$$\begin{bmatrix} 1 & 2 & x_1 \\ 0 & 1 & \frac{1}{5}(2x_1 - x_2) \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 2R_2} \begin{bmatrix} 1 & 0 & \frac{1}{5}(x_1 - 2x_2) \\ 0 & 1 & \frac{1}{5}(2x_1 - x_2) \\ 0 & 0 & 0 \end{bmatrix} \to \frac{1}{5}(x_1 - 2x_2)\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \frac{1}{5}(2x_1 - x_2)\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \dots = \begin{bmatrix} x_1 \\ x_2 \\ x_1 \end{bmatrix}$$ ✓