MAT2705-04/05 20F Final Exam Answers

a) $x_1 = 7\cos t + \cos\sqrt{7}t - \cos 2t$

$x_2 = 7\cos t - 2\cos\sqrt{7}t - 4\cos 2t$

b) $\omega_1 = 1 < \omega_2 = \sqrt{7} \approx 2.646$, ω_3

$T_i = \frac{2\pi}{\omega_i}$: $T_1 = 2\pi \approx 6.283$, $T_2 = 2\pi/\sqrt{7} \approx 2.375$, $T_3 = \frac{2\pi}{2} = \pi \approx 3.142$

c) $$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}'' = \begin{bmatrix} -3 & 2 \\ 4 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 9\cos 2t \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} -3 & 2 \\ 4 & -5 \end{bmatrix}, \quad \vec{F} = \begin{bmatrix} 9\cos 2t \\ 0 \end{bmatrix}$$

d) $$B_{maple} = \begin{bmatrix} 1 & -1/2 \\ 1 & 1 \end{bmatrix} = \langle b_1 | b_2 \rangle$$

$\lambda_1 = -1 \quad \lambda_2 = -7$

e) $$0 = |A - \lambda I| = \begin{vmatrix} -3-\lambda & 2 \\ 4 & -5-\lambda \end{vmatrix}$$

$= (\lambda+3)(\lambda+5) - 8$

$= \lambda^2 + 8\lambda + 15 - 8 = \lambda^2 + 8\lambda + 7 = 0$

$\lambda = \frac{-8 \pm \sqrt{64 - 4(7)}}{2} = \frac{-8 \pm \sqrt{36}}{2} = \frac{-8 \pm 6}{2}$

$= -\frac{2}{2}, \frac{-14}{2} = -1, -7 = \lambda_1, \lambda_2$

$$\lambda = -1: \quad A + I = \begin{bmatrix} -3+1 & 2 \\ 4 & -5+1 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 4 & -4 \end{bmatrix}$$

$$\xrightarrow[\text{obvious}]{\text{rref}} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to x_1 = x_2 = t, \quad x_2 = t \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{b_1}$$

L F

$$\lambda = -7: \quad A + 7I = \begin{bmatrix} -3+7 & 2 \\ 4 & -5+7 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 4 & 2 \end{bmatrix}$$

$$\xrightarrow[\text{obvious}]{\text{rref}} \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to x_1 = -\tfrac{1}{2}x_2 = -\tfrac{1}{2}t, \quad x_2 = t \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$$

L F

$$B = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \longrightarrow B^{-1} = \frac{1}{2+1}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \quad \text{choose (double)} \ b_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

e) $$A_B = B^{-1}AB \overset{\text{Maple}}{=} \begin{bmatrix} -1 & 0 \\ 0 & -7 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \text{ as it should}$$

f) $$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \propto \begin{bmatrix} 1 \\ 1 \end{bmatrix} \to x_2 = x_1 \to m = 1$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \propto \begin{bmatrix} -1 \\ 2 \end{bmatrix} \to x_2 = -2x_1 \to m = -2$$

g) $$\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = B^{-1}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 7 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 14+1 \\ -7+1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 15 \\ -6 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

$$\vec{F}(0) = \begin{bmatrix} 9 \\ 0 \end{bmatrix}, \quad B^{-1}\vec{F}(0) = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 9 \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 18 \\ -9 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$$

h) $$\vec{x}(0) = \begin{bmatrix} 7 \\ 1 \end{bmatrix} = 5\vec{b}_1 - 2\vec{b}_2 \Leftrightarrow \text{sides of projection parallelogram}$$

i) $$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}'' = \begin{bmatrix} -1 & 0 \\ 0 & -7 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \cos 2t\begin{bmatrix} 6 \\ -3 \end{bmatrix} = \begin{bmatrix} -y_1 + 6\cos 2t \\ -7y_2 - 3\cos 2t \end{bmatrix}$$

$y_1'' + y_1 = 6\cos 2t \qquad y_{1h} = c_1\cos t + c_2\sin t$

$y_2'' + 7y_2 = -3\cos 2t \qquad y_{2h} = c_3\cos\sqrt{7}t + c_4\sin\sqrt{7}t$

$y_{1p} = c_5\cos 2t, \quad y_{1p}'' + y_{1p} = (-4+1)c_5\cos 2t = 6\cos 2t$

$y_{2p} = c_6\cos 2t \quad y_{2p}'' + 7y_{2p} = (-4+7)c_6\cos 2t = -3\cos 2t$

$-3c_5 = 6 \to c_5 = -2 \qquad y_{1p} = -2\cos 2t$

$3c_6 = -3 \to c_6 = -1 \qquad y_{2p} = -\cos 2t$

$$\boxed{\begin{aligned} y_1 &= c_1\cos t + c_2\sin t - 2\cos 2t \\ y_2 &= \underbrace{c_3\cos\sqrt{7}t + c_4\sin\sqrt{7}t}_{y_{ih}} \underbrace{- \cos 2t}_{y_{ip}} \end{aligned}}$$

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j) $$\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}1&-1\\1&2\end{bmatrix}\begin{bmatrix}c_1\cos t + c_2\sin t & -2\cos 2t\\ c_3\cos\sqrt{7}t + c_4\sin\sqrt{7}t & -\cos 2t\end{bmatrix}$$

$$\begin{bmatrix}x_1(0)\\x_2(0)\end{bmatrix} = \begin{bmatrix}1&-1\\1&2\end{bmatrix}\begin{bmatrix}c_1-2\\c_3-1\end{bmatrix} = \begin{bmatrix}7\\1\end{bmatrix} \to \begin{bmatrix}c_1-2\\c_3-1\end{bmatrix} = \frac{1}{3}\begin{bmatrix}2&1\\-1&1\end{bmatrix}\begin{bmatrix}7\\1\end{bmatrix} = \begin{bmatrix}5\\-2\end{bmatrix} \text{ (part g)}$$

$\hookrightarrow$ $c_1 = 5+2 = 7$
$c_3 = -2+1 = -1$

$$\begin{bmatrix}x_1'\\x_2'\end{bmatrix} = \begin{bmatrix}1&-1\\1&2\end{bmatrix}\begin{bmatrix}-c_1\sin t + c_2\cos t & +4\sin 2t\\ -\sqrt{7}c_3\sin\sqrt{7}t + \sqrt{7}c_4\cos\sqrt{7}t & +2\sin 2t\end{bmatrix}$$

$$\begin{bmatrix}x_1'(0)\\x_2'(0)\end{bmatrix} = \begin{bmatrix}1&-1\\1&2\end{bmatrix}\begin{bmatrix}c_2\\\sqrt{7}c_4\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \to \begin{bmatrix}c_2\\\sqrt{7}c_4\end{bmatrix} = B^{-1}\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \to \begin{matrix}c_2=0\\c_4=0\end{matrix}$$

Note: $y_{1h} = 7\cos t$
$y_{2h} = -\cos\sqrt{7}t$

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}1&-1\\1&2\end{bmatrix}\begin{bmatrix}7\cos t & -2\cos 2t\\ -\cos\sqrt{7}t & -\cos 2t\end{bmatrix}$$

$$= \begin{bmatrix}7\cos t + \cos\sqrt{7}t & +(-2\cos 2t + \cos 2t)\\ 7\cos t - 2\cos\sqrt{7}t & +(-2\cos 2t - 2\cos 2t)\end{bmatrix}$$

$$= \begin{bmatrix}7\cos t + \cos\sqrt{7}t & -\cos 2t\\ 7\cos t - 2\cos\sqrt{7}t & -4\cos 2t\end{bmatrix}$$

$$\boxed{\begin{aligned}x_1 &= 7\cos t + \cos\sqrt{7}t - \cos 2t\\ x_2 &= 7\cos t - 2\cos\sqrt{7}t - 4\cos 2t\end{aligned}}$$

Maple agrees!

k) $$\begin{bmatrix}x_1\\x_2\end{bmatrix} = \underbrace{7\cos t\begin{bmatrix}1\\1\end{bmatrix} - \cos\sqrt{7}t\begin{bmatrix}-1\\2\end{bmatrix}}_{\vec{x}_h} + \underbrace{\cos 2t\underbrace{\begin{bmatrix}-1\\-4\end{bmatrix}}_{\vec{a}_3}}_{\vec{x}_p}$$

$\leftarrow$ components same sign so tandem mode

response mode

$$\vec{a}_1 = \begin{bmatrix}7\\7\end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix}1\\-2\end{bmatrix}$$

parallelogram 4 corners: $\longrightarrow$

$$\vec{a}_1 + \vec{a}_2 = \begin{bmatrix}7+1\\7-2\end{bmatrix} = \begin{bmatrix}8\\5\end{bmatrix}$$

$$\vec{a}_1 - \vec{a}_2 = \begin{bmatrix}7-1\\7+2\end{bmatrix} = \begin{bmatrix}6\\9\end{bmatrix}$$

$$-\vec{a}_1 - \vec{a}_2 = \begin{bmatrix}-7-1\\-7+2\end{bmatrix} = \begin{bmatrix}-8\\-5\end{bmatrix}$$

$$-\vec{a}_1 + \vec{a}_2 = \begin{bmatrix}-7+1\\-7-2\end{bmatrix} = \begin{bmatrix}-6\\-9\end{bmatrix}$$

l) see Maple for the $t = 0..200$ plot of $\vec{x}_h$, which fits nicely in this box.

in pencil (pen?). Are the part g) new components numerically consistent with your plot parallelogram? Explain.

i) Find by hand the general solution of the corresponding decoupled system of DEs $\vec{y}\,'' = \Lambda_B \vec{y} + B^{-1}\vec{F}$. First write these equations out in explicit matrix form, then obtain the two equivalent scalar DEs which are its components. Then solve them to find their general solutions using the method of undetermined coefficients. State your general solutions in scalar form and box them: $y_1(t) = ..., y_2(t) = ...$, identifying the homogeneous and particular parts of each solution: $y_1 = y_{1h} + y_{1p}, y_2 = y_{2h} + y_{2p}$.

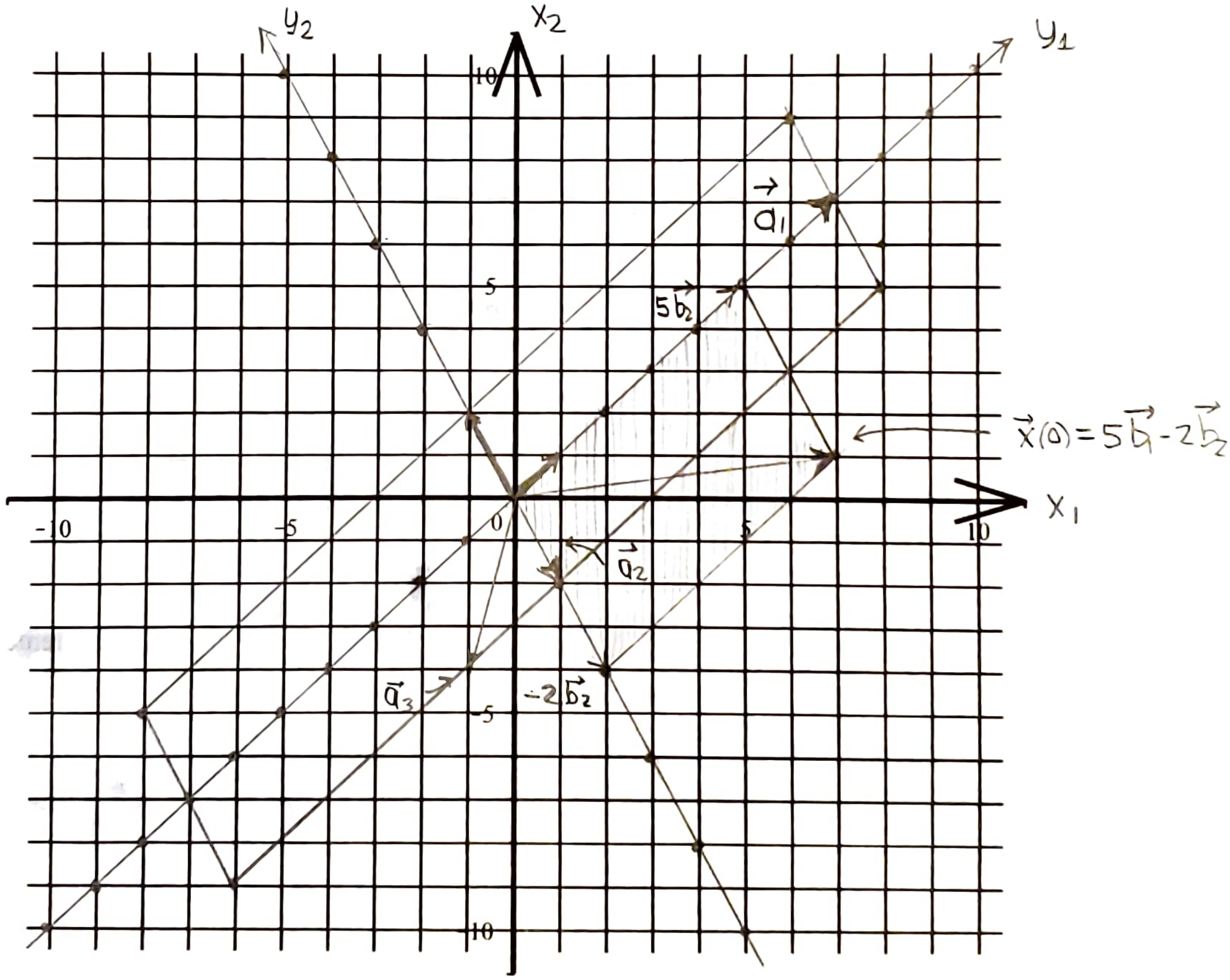
j) Then express the general solution for $\vec{x} = B\vec{y}$ in explicit matrix form (without multiplying matrix factors) ***and impose the initial conditions*** using matrix methods to solve the linear systems. Write out and box the final scalar solutions: $x_1(t) = ..., x_2(t) = ...$. Do they agree with Maple's solution from part a)? If not, look for your error. Did you input the equations correctly?

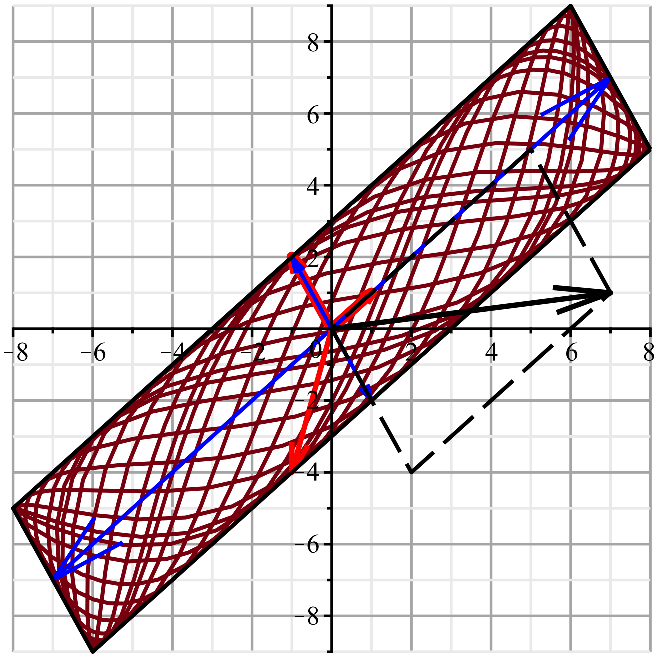
k) Express the (correct) solution as a sum of the two eigenvector modes and the response mode in the form:

$\vec{x} = y_{1h}\vec{b}_1 + y_{2h}\vec{b}_2 + \cos(2t)\,\vec{a}_3$

thus identifying the particular solution $\vec{x}_p$ (last term), the response vector coefficient $\vec{a}_3$ and the homogeneous solution $\vec{x}_h$ (first two terms), as well as the final expressions for the two decoupled variables y_{1h} and y_{2h} . Is the response term a tandem or accordian mode? Identify the vectors $\vec{a}_1 = y_{1h}(0)\,\vec{b}_1, \vec{a}_2 = y_{2h}(0)\,\vec{b}_2$. Include these vectors in your plot (together with $\vec{a}_3$) and use them to create the sides of the bounding box enclosing the solution with four endpoints $\pm\vec{a}_1 \pm \vec{a}_2$.

l) Plot the homogeneous solution for large t to "see" the parallelogram box containing it. Is it consistent with your plot on the grid?

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