MAT2705-04/05 20F Test 3 Answers (1)

① a) $4x''+4x'+101x=0$

$x''+x'+\frac{101}{4}x=0$

$k_0=1$, $\tau_0=1$, $Q=\omega_0=\frac{\sqrt{101}}{2}\approx 5.025$

$T_0=\frac{2\pi}{\omega_0}=\frac{4\pi}{\sqrt{101}}\approx 1.250$

$x=e^{rt}$: $\underbrace{(4r^2+4r+101)}_{=0}e^{rt}=0$

$r=\frac{-4\pm\sqrt{16-4(4)(101)}}{2(4)}=\frac{-1\pm\sqrt{-100}}{2}$

$=-\frac{1}{2}\pm i\,\frac{10}{2}=-\frac{1}{2}\pm 5i$

$k_1=\frac{1}{2}$, $\omega_1=5$

$\tau_1=2$, $T_1=\frac{2\pi}{5}\approx 1.257$

$e^{rt}=e^{-\frac{1}{2}t}e^{\pm i5t}=e^{-\frac{1}{2}t}(\cos 5t\pm i\sin 5t)$

$\hookrightarrow$ real basis: $e^{-\frac{t}{2}}\cos 5t,\ e^{-\frac{t}{2}}\sin 5t$

$x=e^{-t/2}(c_1\cos 5t+c_2\sin 5t)$ gen soln

$x'=-\frac{1}{2}e^{-t/2}(c_1\cos 5t+c_2\sin 5t)$
$\quad +e^{-t/2}(-5c_1\sin 5t+5c_2\cos 5t)$

$x(0)=c_1=10$

$x'(0)=-\frac{1}{2}c_1+5c_2=5$

$c_2=\frac{5+\frac{1}{2}(10)}{5}=\frac{10}{5}=2$

$$\boxed{x=e^{-t/2}(10\cos 5t+2\sin 5t)}$$

IVP soln

b) $x'=-\frac{1}{2}e^{-t/2}(10\cos 5t+2\sin 5t)$
$\quad +e^{-t/2}(-50\sin 5t+10\cos 5t)=0$

$(-5+10)\cos 5t+(-1-50)\sin 5t=0$

$5\cos 5t-51\sin 5t=0$

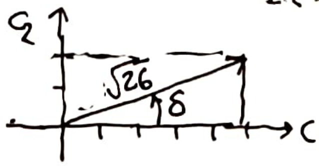
$\tan 5t=\frac{5}{51}\to 5t=\arctan\frac{5}{51}+\pi$

the first critical pt is a local max, the second is half cycle later in time and it is the global minimum as the plot shows.

b) continued $t=\frac{1}{5}(\arctan\frac{5}{51}+\pi)\approx 0.6479$

$x\approx -7.3496$ (Maple)

c) $x_{sinusoidal}=10\cos 5t+2\sin 5t$
$\quad = 2(5\cos 5t+\sin 5t)$



$\tan\delta=\frac{1}{5}$

$\delta=\arctan\frac{1}{5}\approx 0.197$
$\approx .031$ cycles
≈ 11.3 degrees

$A=2\sqrt{26}$
≈ 10.198

d) $x_{env}=\pm A e^{-t/2}$

decay window: $t=0..5\tau_1=0..10$

e) $101\,[\ x_p=c_3\cos\omega t+c_4\sin\omega t$

$4\,[\ x_p'=-c_3\omega\sin\omega t+c_4\omega\cos\omega t$

$4\,[\ x_p''=-c_3\omega^2\cos\omega t-c_4\omega^2\sin\omega t$

$4x_p''+4x_p'+101x_p$
$=[(101-4\omega^2)c_3+4\omega c_4]\cos\omega t$
$+[-4\omega c_3+(101-4\omega^2)c_4]\sin\omega t$
$=2\cos\omega t+\sin\omega t$

$\begin{bmatrix}101-4\omega^2 & 4\omega\\ -4\omega & 101-4\omega^2\end{bmatrix}\begin{bmatrix}c_3\\ c_4\end{bmatrix}=\begin{bmatrix}2\\ 1\end{bmatrix}\longrightarrow$

$\begin{bmatrix}c_3\\ c_4\end{bmatrix}=\frac{1}{(101-4\omega^2)^2+16\omega^2}\begin{bmatrix}101-4\omega^2 & -4\omega\\ 4\omega & 101-4\omega^2\end{bmatrix}\begin{bmatrix}2\\ 1\end{bmatrix}$

$\mathcal{D}$

$=\frac{1}{\mathcal{D}}\begin{bmatrix}2(101-4\omega^2)-4\omega\\ 8\omega+(101-4\omega^2)\end{bmatrix}$

$\mathcal{D}=(101-4\omega^2)^2+16\omega^2$
$=16\omega^2-792\omega^2+10201$ (Maple)

$x_p=\frac{1}{\mathcal{D}}\left[(202-8\omega^2-4\omega)\cos\omega t+(8\omega+101-4\omega^2)\sin\omega t\right]$

steady state soln.

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1 e) continued

$$A(\omega) = \frac{\sqrt{(202 - 8\omega^2 - 4\omega)^2 + (8\omega + 101 - 4\omega^2)^2}}{\mathcal{D}}$$

$$\underset{\text{Maple}}{=} \frac{\sqrt{5}}{(16\omega^4 - 792\omega^2 + 10201)^{1/2}} = \frac{\sqrt{5}}{\mathcal{D}}$$

$$A(0) = \sqrt{\frac{5}{10201}} = \frac{\sqrt{5}}{101}$$

$$\frac{A(\omega)}{A(0)} = \frac{101}{(16\omega^4 - 792\omega^2 + 10201)^{1/2}}$$

$$A'(\omega) = \sqrt{5}\left(-\tfrac{1}{2}\right)\mathcal{D}^{-3/2}\underbrace{(64\omega^3 - 2(792\omega))}_{= 64\omega\left(\omega^2 - \frac{99}{4}\right)} = 0$$

$\to \omega = 0, \sqrt{99}/2$

$$\omega_{peak} = \frac{\sqrt{99}}{2} = \frac{3\sqrt{11}}{2} \approx 4.9749$$

$$\lesssim \omega_0 \approx 5.025$$

close!

$$\frac{A(\omega_{max})}{A(0)} \underset{\text{Maple}}{=} \frac{101}{20} = 5.0500 \gtrsim Q \approx 5.0250$$

very close!

plot roughly $t = 0..5\omega_0 \approx \to 0..25$

MAT2705-04/05 Test 3 20F Answers (3)

② a) $\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \underbrace{\begin{bmatrix} -1/2 & 0 & 1/2 \\ 1/2 & -1/5 & 0 \\ 0 & 1/5 & -1/2 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 18 \\ 0 \\ 0 \end{bmatrix}$

b) $0 = |A - \lambda I| = \begin{vmatrix} -1/2-\lambda & 0 & 1/2 \\ 1/2 & -1/5-\lambda & 0 \\ 0 & 1/5 & -1/2-\lambda \end{vmatrix}$

$\overset{\text{Maple}}{=} -\lambda^3 - \frac{6}{5}\lambda^2 - \frac{9}{20}\lambda$

$= -\lambda\left(\lambda^2 + \frac{6}{5}\lambda + \frac{9}{20}\right)$

$\lambda = 0, \; \lambda = \dfrac{-\frac{6}{5} \pm \sqrt{\frac{36}{25} - 4\left(\frac{9}{20}\right)}}{2} = \dfrac{-\frac{6}{5} \pm \sqrt{-\frac{9}{25}}}{2}$

$= -\frac{3}{5} \pm \frac{1}{2} \cdot i\left(\frac{3}{5}\right) = -\frac{3}{5} \pm \frac{3}{10} i$

$\lambda_1 = 0, \quad \lambda_2 = -\frac{3}{5} + \frac{3i}{10}, \quad \lambda_3 = -\frac{3}{5} - \frac{3i}{10}$

$\underline{\lambda = 0}$: $A = \begin{bmatrix} -1/2 & 0 & 1/2 \\ 1/2 & -1/5 & 0 \\ 0 & 1/5 & -1/2 \end{bmatrix} \rightarrow \begin{bmatrix} -1/2 & 0 & 1/2 \\ 0 & -1/5 & 1/2 \\ 0 & 1/5 & -1/2 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -5/2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} \rightarrow x_1 = t \\ \rightarrow x_2 = 5/2\, t \\ \\ \end{matrix}$

L L F ⟶ $x_3 = t$

$\langle x_1, x_2, x_3 \rangle = t \underbrace{\langle 1, 5/2, 1 \rangle}_{b_1}$

$\underline{\lambda = -\frac{3}{5} + \frac{i3}{10}}$: $\begin{bmatrix} -\frac{1}{2} + \frac{3}{5} - \frac{3i}{10} & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{5} + \frac{3}{5} - \frac{3i}{10} & 0 \\ 0 & \frac{1}{5} & -\frac{1}{2} + \frac{3}{5} - \frac{3i}{10} \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{10} - \frac{3i}{10} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{2}{5} - \frac{3i}{10} & 0 \\ 0 & \frac{1}{5} & \frac{1}{10} - \frac{3i}{10} \end{bmatrix} \xrightarrow[\text{Maple}]{\text{rref}} \begin{bmatrix} 1 & 0 & \frac{1}{2} + \frac{3i}{2} \\ 0 & 1 & \frac{1}{2} - \frac{3i}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

L L F

b) continued

$x_3 = t, \quad x_1 = -\left(\frac{1}{2} + \frac{3i}{2}\right)t$

$x_2 = -\left(\frac{1}{2} - \frac{3i}{2}\right)t$

$\langle x_1, x_2, x_3 \rangle = t \underbrace{\left\langle -\frac{1}{2} - \frac{3i}{2}, -\frac{1}{2} + \frac{3i}{2}, 1 \right\rangle}_{b_2 = \overline{b_3}}$

$\therefore B = \begin{bmatrix} 1 & -\frac{1}{2} - \frac{3i}{2} & -\frac{1}{2} + \frac{3i}{2} \\ 5/2 & -\frac{1}{2} + \frac{3i}{2} & -\frac{1}{2} - \frac{3i}{2} \\ 1 & 1 & 1 \end{bmatrix}$

d) $x = By, \quad y = B^{-1}x$

$x' = Ax$

$B^{-1}(By)' = B^{-1}A(By)$

$y' = B^{-1}ABy = A_B\, y$

$A_B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{3}{5} + \frac{3i}{10} & 0 \\ 0 & 0 & -\frac{3}{5} - \frac{3i}{10} \end{bmatrix}$

$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = A_B \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \left(-\frac{3}{5} + \frac{3i}{10}\right) y_2 \\ \left(-\frac{3}{5} - \frac{3i}{10}\right) y_3 \end{bmatrix}$ decoupled

$y_1' = 0 \rightarrow y_1 = c_1$

$y_2' = \left(-\frac{3}{5} + \frac{3i}{10}\right) y_2 \rightarrow y_2 = c_2 e^{-\frac{3t}{5} + \frac{i3}{10}t}$

$y_3' = \left(-\frac{3}{5} - \frac{3i}{10}\right) y_3 \rightarrow y_3 = \underline{c_3} e^{-\frac{3t}{5} - \frac{3it}{10}}$

$= \overline{c_2}$ for real solns

$\therefore x = c_1 b_1 + \underbrace{c_2 e^{-\frac{3t}{5}} e^{3it/10} b_2 + \text{c.c.}}$

↳ replace by explicitly real soln

MAT2705-04/05 20F Test 3 Answers (4)

$$\boxed{\begin{aligned} x_1 &= 4 + e^{-3t/5}\left(14\cos\tfrac{3t}{10} - 2\sin\tfrac{3t}{10}\right) \\ x_2 &= 10 + e^{-3t/5}\left(-10\cos\tfrac{3t}{10} + 10\sin\tfrac{3t}{10}\right) \\ x_3 &= 4 + e^{-3t/5}\left(-4\cos\tfrac{3t}{10} - 8\sin\tfrac{3t}{10}\right) \end{aligned}}$$

$\tau = \frac{5}{3} \sim 1.67$

$5\tau \approx 8.33 \to 10$ (nice even #)

plotwindow $t = 0..10$

② e)

$$e^{-\frac{3t}{5}} e^{3ti/10}\, b_2$$

$$= e^{-3t/5}\left(\cos\tfrac{3t}{10} + i\sin\tfrac{3t}{10}\right)\begin{bmatrix} -\frac{1}{2} - \frac{3i}{2} \\ -\frac{1}{2} + \frac{3i}{2} \\ 1 \end{bmatrix}$$

$$= e^{-3t/5}\begin{bmatrix} \frac{1}{2}(-c + 3s + i(-3c - s)) \\ \frac{1}{2}(-c - 3s + i(3c - s)) \\ c + i\, s \end{bmatrix}$$

$$= \underbrace{e^{-3t/5}\begin{bmatrix} \frac{1}{2}(-c+3s) \\ \frac{1}{2}(-c-3s) \\ c \end{bmatrix}}_{X_1} + i\, \underbrace{e^{-3t/5}\begin{bmatrix} \frac{1}{2}(-3c-s) \\ \frac{1}{2}(3c-s) \\ s \end{bmatrix}}_{X_2}$$

$$x = c_1 b_1 + c_2 X_1 + c_2 X_2$$

f)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 5/2 \\ 1 \end{bmatrix} + c_2 e^{-3t/5}\begin{bmatrix} \frac{1}{2}(-\cos\frac{3t}{10} + 3\sin\frac{3t}{10}) \\ \frac{1}{2}(-\cos\frac{3t}{10} - 3\sin\frac{3t}{10}) \\ \cos\frac{3t}{10} \end{bmatrix} + c_3 e^{-3t/5}\begin{bmatrix} \frac{1}{2}(-3\cos\frac{3t}{10} - \sin\frac{3t}{10}) \\ \frac{1}{2}(3\cos\frac{3t}{10} - \sin\frac{3t}{10}) \\ \sin\frac{3t}{10} \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 5/2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_3\begin{bmatrix} -\frac{3}{2} \\ \frac{3}{2} \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1/2 & -3/2 \\ 5/2 & -1/2 & 3/2 \\ 1 & 1 & 0 \end{bmatrix}}_{B}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 18 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = B^{-1}\begin{bmatrix} 18 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 2 & 2 & 2 \\ -2 & -2 & 7 \\ -4 & 2 & -1 \end{bmatrix}\begin{bmatrix} 18 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ -8 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 5/2 \\ 1 \end{bmatrix} - 4e^{-3t/5}\begin{bmatrix} \frac{1}{2}(-c+3s) \\ \frac{1}{2}(-c-3s) \\ c \end{bmatrix} - 8e^{-3t/5}\begin{bmatrix} \frac{1}{2}(-3c-s) \\ \frac{1}{2}(3c-s) \\ s \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 10 \\ 4 \end{bmatrix} + e^{-3t/5}\begin{bmatrix} (2+12)c + (-6+4)s \\ (2-12)c + (6+4)s \\ -4c - 8s \end{bmatrix} = \begin{bmatrix} 4 + e^{-3t/5}(14\cos\frac{3t}{10} - 2\sin\frac{3t}{10}) \\ 10 + e^{-3t/5}(-10\cos\frac{3t}{10} + 10\sin\frac{3t}{10}) \\ 4 + e^{-3t/5}(-4\cos\frac{3t}{10} - 8\sin\frac{3t}{10}) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

MAT2705-04/05 20F Test 3 Answers (5)

3 a) $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \underbrace{\begin{bmatrix} -10 & 2 \\ 3 & -15 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \end{bmatrix}$

Not necessary but for fun:

$0 = |A - \lambda I| = \begin{vmatrix} -10-\lambda & 2 \\ 3 & -15-\lambda \end{vmatrix} = (\lambda+10)(\lambda+15) - 6$

$= \lambda^2 + 25\lambda + 144$

$\lambda = \dfrac{-25 \pm \sqrt{625 - 4(144)}}{2} = \dfrac{-25 \pm 7}{2} = -16, -9$

$\lambda_1 = -16, \quad \lambda_2 = -9$

$\underline{\lambda = -16}$: $A + 16I = \begin{bmatrix} -10+16 & 2 \\ 3 & -15+16 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}$

$\to \underset{\text{LF}}{\begin{bmatrix} 1 & 1/3 \\ 0 & 0 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_2 = t, \quad x_1 = -\frac{1}{3} t \quad \langle x_1, x_2 \rangle = t \underbrace{\langle -\tfrac{1}{3}, 1 \rangle}_{b_1}$

$\lambda = -9$: $A + 9I = \begin{bmatrix} -10+9 & 2 \\ 3 & -15+9 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$

$\to \underset{\text{LF}}{\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_2 = t, \quad x_1 = 2t \quad \langle x_1, x_2 \rangle = t \underbrace{\langle 2, 1 \rangle}_{b_2}$

$B = \langle b_1 | b_2 \rangle = \begin{bmatrix} -\frac{1}{3} & 2 \\ 1 & 1 \end{bmatrix}$

$\lambda = -16, -9$

} Maple result you can start here)

$B^{-1} = \dfrac{1}{-\frac{1}{3} - 2} \begin{bmatrix} 1 & -2 \\ -1 & -1/3 \end{bmatrix} = \dfrac{1}{7} \begin{bmatrix} -3 & 6 \\ 3 & 1 \end{bmatrix}$

$A_B = B^{-1} A B = \begin{bmatrix} -16 & 0 \\ 0 & -9 \end{bmatrix}$

b) $x = By, \quad y = B^{-1} x$

$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \dfrac{1}{7} \begin{bmatrix} -3 & 6 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ -3 \end{bmatrix} = \dfrac{1}{7} \begin{bmatrix} -24 - 18 \\ 24 - 3 \end{bmatrix}$

$= \begin{bmatrix} -6 \\ 3 \end{bmatrix}$

c) so $\vec{x}(0) = -6 \vec{b}_1 + 3 \vec{b}_2$

$B^{-1}(By') = B^{-1} A (By)$

$y' = A_B y$

$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} -16 & 0 \\ 0 & -9 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -16 y_1 \\ -9 y_2 \end{bmatrix}$

$y_1' = -16 y_1, \quad y_1 = c_1 e^{-16t}$

$y_2' = -9 y_2, \quad y_2 = c_2 e^{-9t}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1/3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{-16t} \\ c_2 e^{-9t} \end{bmatrix}$

$\begin{bmatrix} 8 \\ -3 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = B \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \to \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = B^{-1} \begin{bmatrix} 8 \\ -3 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -6 e^{-16t} \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix} + 3 e^{-9t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 e^{-16t} + 6 e^{-9t} \\ -6 e^{-16t} + 3 e^{-9t} \end{bmatrix}}$$

e) $\tau_1 = \frac{1}{16}, \quad \tau_2 = \frac{1}{9} > \tau_1$

τ_2 larger, $\quad 5\tau_2 \sim 5/9$

need $t = 0..1$ (easy #) in DEplot.