# MAT 2705-04/05 20F test2 Answers (1)

① a)
$$4x_1 + 3x_2 + 3x_3 = 1$$
$$5x_1 + 6x_2 + 3x_3 = 2$$
$$3x_1 + 5x_2 + 2x_3 = 3$$

$$\underbrace{\begin{bmatrix} 4 & 3 & 3 \\ 5 & 6 & 3 \\ 3 & 5 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{b}$$

$$x = A^{-1} b = \begin{bmatrix} 3 & -4 & 3 \\ 1 & -2 & 2 \\ -7 & 11 & -9 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

scalar solns

$$= \begin{bmatrix} 3-8+9 \\ 1-4+6 \\ -7+22-27 \end{bmatrix} = \boxed{\begin{bmatrix} 4 \\ 3 \\ -12 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} \rightarrow \boxed{\begin{matrix} x_1 = 4 \\ x_2 = 3 \\ x_3 = -12 \end{matrix}}$$

b) $\det(A) \underset{\text{Maple}}{=} -1 \neq 0$

This guarantees A row reduces to the identity and so has an inverse, and will produce integer solns from integer b.

② $Ax = b$; $C = \langle A | b \rangle$

a)
$$C = \begin{bmatrix} 1 & 1 & -1 & 7 & 1 \\ 1 & 4 & 5 & 16 & 10 \\ 1 & 3 & 3 & 13 & 7 \\ 2 & 5 & 4 & 23 & 11 \end{bmatrix} \xrightarrow[\text{maple}]{\text{rref}}$$

$$\begin{array}{c} \begin{bmatrix} 1 & 0 & -3 & 4 & -2 \\ 0 & 1 & 2 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} x_1 & x_2 & x_3 & x_4 & \\ L & L & F & F & \end{matrix} \end{array} \begin{matrix} \rightarrow x_1 - 3x_3 + 4x_4 = -2 \\ \rightarrow x_2 + 2x_3 + 3x_4 = 3 \\ \rightarrow 0 = 0 \end{matrix}$$

$x_3 = t_1$, $x_4 = t_2$

$$x_1 = -2 + 3t_1 - 4t_2$$
$$x_2 = 3 - 2t_1 - 3t_2$$

②a)
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 + 3t_1 - 4t_2 \\ 3 - 2t_1 - 3t_2 \\ t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 0 \\ 0 \end{bmatrix} + \underbrace{t_1 \overset{e_1}{\begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix}} + t_2 \overset{e_2}{\begin{bmatrix} -4 \\ -3 \\ 0 \\ 1 \end{bmatrix}}}_{\text{homogeneous soln: spans 2-d subspace of } \mathbb{R}^4}$$

b) $e_1$: $3\vec{v}_1 - 2\vec{v}_2 + \vec{v}_3 = \vec{0}$

$e_2$: $-4\vec{v}_1 - 3\vec{v}_2 + \vec{v}_4 = \vec{0}$

2 vectors are linearly independent. The row reduction algorithm picks out the first two (leading) columns.

c) Setting $t_1 = t_2 = 0$ gives the unique linear combination of $\vec{v}_1$ and $\vec{v}_2$:

$$\begin{bmatrix} 1 \\ 10 \\ 7 \\ 11 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 4 \\ 3 \\ 5 \end{bmatrix} = \underbrace{\begin{bmatrix} -2+3 \\ -2+12 \\ -2+9 \\ -4+15 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ 7 \\ 11 \end{bmatrix}}_{\text{check!}}$$

③ b) $\vec{v}_3 = y_1 \vec{v}_1 + y_2 \vec{v}_2 \leftrightarrow \langle v_1 | v_2 \rangle \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = v_3$

$$\underbrace{\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}}_{A} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underbrace{\frac{1}{3+4} \begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix}}_{A^{-1}} \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$= \frac{1}{7} \begin{bmatrix} 12+2 \\ -8+1 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 14 \\ -7 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \text{ so}$$

$$\boxed{\begin{bmatrix} 4 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} -2 \\ 3 \end{bmatrix}} = \underbrace{\begin{bmatrix} 2+2 \\ 4-3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}}_{\text{check}}$$

agrees with graph!

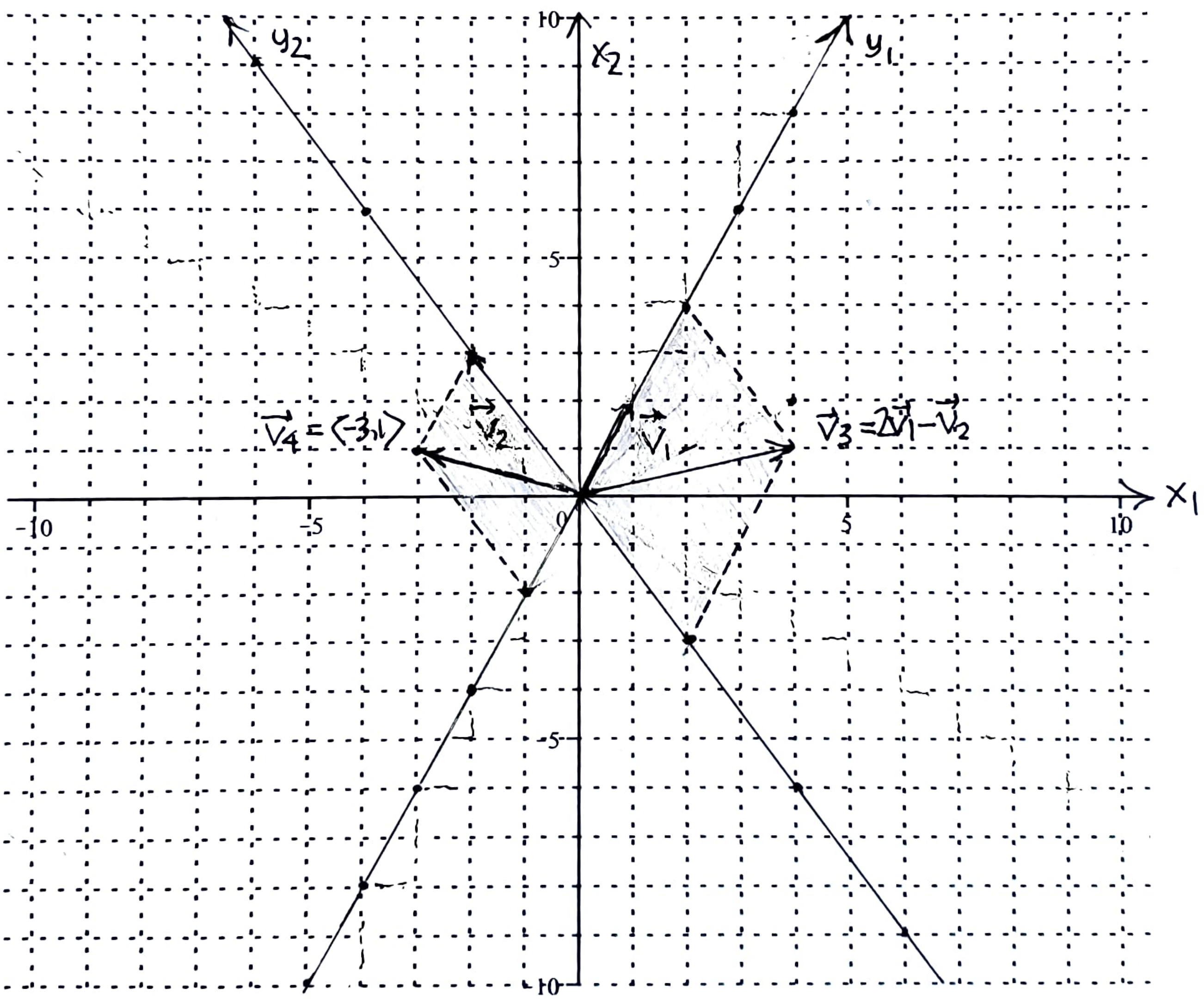
c) $-\vec{v}_1 + \vec{v}_2 = -\langle 1,2 \rangle + \langle -2,3 \rangle = \langle -1-2, -2+3 \rangle$
$= \langle -3, 1 \rangle$

3. a) On the grid below, **draw in** arrows representing the vectors $\vec{v_1} = \langle 1, 2 \rangle$ and $\vec{v_2} = \langle -2, 3 \rangle$ and $\vec{v_3} = \langle 4, 1 \rangle$ and **label** them by their symbols. **Extend** the basis vectors $\{\vec{v_1}, \vec{v_2}\}$ to the corresponding coordinate axes for $(y_1, y_2)$ and **mark** the positive direction with an arrow head and the axis label. Mark off tickmarks on these axes for integer values of the new coordinates. Then **draw in** the parallelogram with edges parallel to the new axes for which $\vec{v_3}$ is the main diagonal and shade it in in pencil lightly. Read off the coordinates $(y_1, y_2)$ of $\vec{v_3}$ with respect to these two vectors (write them down) and **express** $\vec{v_3}$ as a linear combination of these vectors; **put this equation** at the tip of this vector.

b) Now use matrix methods to express $\vec{v_3}$ as a linear combination of the other two vectors (show all steps in this process), box it and then check your linear combination by expanding it out. Does your matrix result agree with your graphical result in part a)?

c) **Draw in** the arrow representing the vector $\vec{v_4}$ whose new coordinates are $(y_1, y_2) = (-1, 1)$ and **label** the tip of $\vec{v_4}$ by its symbol. Draw in the projection parallelogram associated with the new coordinates and lightly shade it in in pencil. Determine its old coordinates $(x_1, x_2)$ graphically. Then evaluate them using a linear combination.

3) a)




▶ **solution [put all other work and responses on separate sheets]**