MAT2705-04/05 20F Test 1 Answers (1)

(1) a) $\frac{dP}{dt} = .00225P - .0003P^2$

$= (.00225 - .0003P)P = 0$

$P = \frac{.00225}{.0003} = \boxed{75 \equiv M}$

$\therefore P = 75$ is the equilibrium soln

$\frac{dP}{dt} = .0003P\left(\frac{.00225}{.0003} - P\right)$

$= \underbrace{\underbrace{.0003}_{k}\, P(\underbrace{75}_{M} - P)}$ logistic parameters

$\left(\begin{array}{l} kM = .00225 = 1/\tau \\ \tau \approx 44.4\ldots \end{array} \right)$

b) Maple: $\boxed{P(t) = \frac{75}{1 + 75\, e^{-9t/400} C}}$

gen soln:

c) $\boxed{\tau = \frac{400}{9}} \boxed{\approx 44.4}$

d) $P(t) = -\frac{75 P_0}{e^{-\frac{9t}{400}} P_0 - 75 e^{-9t/400} - P_0}$

$\boxed{= \frac{75 P_0}{P_0 + (P_0 - 75)e^{-9t/400}}}$ cleaning up Maple expression

IVP soln

e) $P_0 = P(0) = \frac{75}{1 + 75Ce^0}$

$P_0 = \frac{75}{1 + 75C} \rightarrow P_0(1 + 75C) = 75$

$P_0 + 75P_0C = 75 \rightarrow C = \frac{75 - P_0}{75P_0}$

$P(t) = \frac{75}{1 + \cancel{75}\frac{(75 - P_0)}{\cancel{75}P_0} e^{-9t/400}} \left(\frac{P_0}{P_0}\right)$

$= \frac{75P_0}{P_0 + (75 - P_0)e^{-9t/400}}$ ✓

f) $P_0 = 25$

$\boxed{\begin{array}{l} P(t) = \frac{75(25)}{25 + \underbrace{(75-25)}_{50} e^{-9t/400}} \quad \frac{1/25}{1/25} \\ = \boxed{\frac{75}{1 + 2e^{-9t/400}}} \end{array}}$

g) $P(5 \cdot 12) = P(60) = \frac{75}{1 + 2e^{-\underbrace{60 \cdot 9/400}_{=27/20}}}$

$\approx 49.391 \approx 49.4$

$\frac{P(60)}{M} \approx 0.6585 \approx 0.659$

$\boxed{\text{about } 66\%}$

h) $\frac{\cancel{75}}{1 + 2e^{-t/\tau}} = .99\,\cancel{75}$

$1 + 2e^{-t/\tau} = \frac{1}{.99}$

$2e^{-t/\tau} = \frac{1}{.99} - 1$

$e^{-t/\tau} = \frac{1}{2}\left(\frac{1}{.99} - 1\right) = \frac{1}{2}\left(\frac{\overbrace{1 - .99}^{.01}}{.99}\right)$

$e^{t/\tau} = 2\,\frac{.99}{.01} = 2.99$

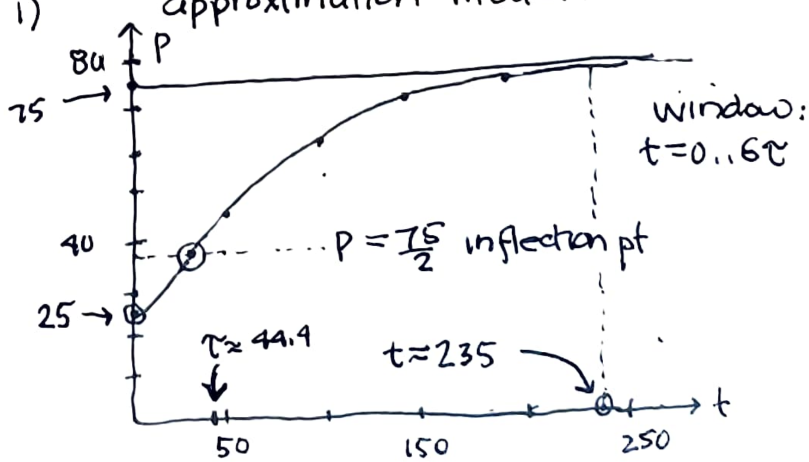
$\frac{t}{\tau} = \underbrace{\ln(2 \cdot 99)}_{5.29 \text{ characteristic times}}$

$t \approx \boxed{235.0 \text{ months}}$

≈ 19.6 years

1% of 75 deer is 3/4 deer! Remember this is a continuous approximation model!

i)



② a) $\frac{dy}{dt} = y - te^{-t}$ linear!

$\frac{dy}{dt} - y = -te^{-t}$ standard form

$e^{\int -1\,dt} = e^{-t}$ integrating factor

$e^{-t}\left(\frac{dy}{dt} - y\right) = e^{-t}(-te^{-t})$

$\frac{d}{dt}\left(e^{-t}y\right) = -te^{-2t}$

$e^{-t}y = -\int te^{-2t}\,dt$

$= \frac{(1+2t)}{4}e^{-2t} + C$ Maple

$$\boxed{\begin{aligned} y &= e^{t}\left(\tfrac{1}{4}(1+2t)e^{-2t} + C\right) \\ &= \tfrac{1}{4}(1+2t)e^{t}e^{-2t} + Ce^{t} \\ &= \boxed{\tfrac{1}{4}(1+2t)e^{-t} + Ce^{t}} \end{aligned}}$$

gen soln, Maple agrees, doesnt multiply thru

b) $\frac{1}{4} = y(0) = \frac{1}{4}(1+2\cdot 0)e^{0} + Ce^{0}$
$= \frac{1}{4} + C$

$C = 0$

$$\boxed{y = \tfrac{1}{4}(1+2t)e^{-t}}$$

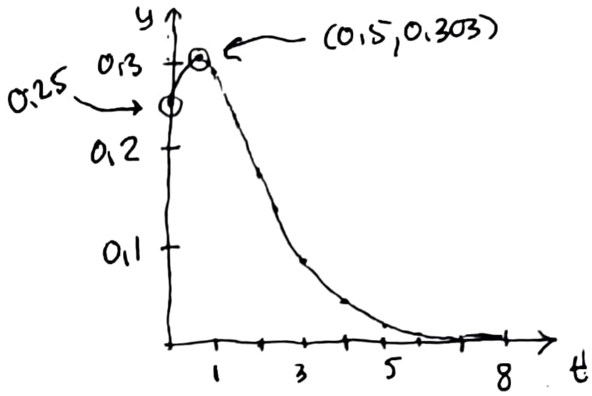
c) $0 = \frac{dy}{dt} = \frac{1}{4}\left[(0+2)e^{-t} + (1+2t)(-e^{-t})\right]$
$= \frac{1}{4}[2 - (1+2t)]e^{-t} = \frac{1}{4}(1-2t)e^{-t}$

$1 - 2t = 0 \rightarrow \boxed{t = 1/2 = 0.5}$

$y(\tfrac{1}{2}) = \frac{1}{4}(1 + 2(\tfrac{1}{2}))e^{-1/2} = \boxed{\tfrac{1}{2}e^{-1/2}} \approx .30326$

$\boxed{\approx 0.303}$

d) pure exponential window:
$\tau = 1$, $t = 0..5\tau = 0..5$
But product te^{-t} delays exponential decay by initial growth.
Trial & error window: $t = 0..8$



MAT 2705-04/05 20F Test 1 Answers (3)

3 a) $\frac{dv}{dt} = -k\, v^{3/2}$ separable

$\int v^{-3/2}\,dv = -\int k\,dt$ separate, integrate

$\frac{v^{-1/2}}{-1/2} = -kt + C_1$ implicit soln

$v^{-1/2} = -\frac{1}{2}(-kt + C_1) = \frac{1}{2}kt + \underbrace{\frac{C_1}{2}}_{\equiv C_1' \ge 0} \ge 0$ (for $t \ge 0$)

$v^{1/2} = \frac{1}{C + \frac{1}{2}kt}$ $(C \ge 0)$

$\boxed{v = \frac{1}{(C + \frac{1}{2}kt)^2}}$ gen soln

easier: $\boxed{\begin{aligned} v_0^{1/2} &= \frac{1}{C} \ge 0 \\ C &= v_0^{-1/2}\end{aligned}}$

$v_0 = \frac{1}{(C+0)^2} \longrightarrow C^2 = \frac{1}{v_0} \to C = \frac{1}{v_0^{1/2}}$

$v = \frac{1}{(v_0^{-1/2} + \frac{1}{2}kt)^2} = \frac{1}{\left(\frac{1}{v_0^{1/2}} + \frac{kt}{2}\right)^2}$

$= \frac{1}{\left(\frac{2 + ktv_0^{1/2}}{2v_0^{1/2}}\right)^2} = \frac{(2v_0^{1/2})^2}{(2 + ktv_0^{1/2})^2}$

$v = \frac{4v_0}{(2 + ktv_0^{1/2})^2}$ ✓

dummy variable

$x = x_0 + \int_0^t v(w)\,dw$

$= x_0 + \underbrace{\int_0^t \frac{4v_0\,dw}{(2 + kv_0^{1/2}w)^2}}$ $= x_0 + \underbrace{-\frac{v_0^{1/2}}{k}\frac{4}{(2 + kv_0^{1/2}w)}\Big|_0^t}$

$\frac{4v_0^{1/2}}{k}\left(\frac{1}{2} - \frac{1}{2 + kv_0^{1/2}t}\right)$

$\boxed{\frac{2v_0^{1/2}}{k}\left(1 - \frac{2}{2 + kv_0^{1/2}t}\right)}$ ✓

$u = 2 + k_0 v_0^{1/2} w$

$\frac{du}{dw} = k_0 v_0^{1/2}$.

$\frac{du}{k_0 v_0^{1/2}} = dw$

$\int \frac{4v_0}{u^2}\frac{du}{k_0 v_0^{1/2}}$

$= \frac{4v_0^{1/2}}{k_0}\int u^{-2}\,du = \frac{4v_0^{1/2}}{k_0}\frac{u^{-1}}{-1} + C$

$= -\frac{4v_0^{1/2}}{k_0(2 + k_0 v_0^{1/2} w)}$

c) $x_\infty = \lim_{t\to\infty}\left(x_0 + \frac{2v_0^{1/2}}{k}\left(1 - \cancel{\frac{2}{2 + kv_0^{1/2}t}}\right)\right)$ $\to 0$

$= x_0 + \underbrace{\frac{2v_0^{1/2}}{k}}_{\Delta x \text{ net displacement}}$

d) $v_0 = 25$

$16 = v(\tfrac{1}{2}) = \frac{4(25)}{(2 + \frac{1}{2}k\underbrace{(25)^{1/2}}_{5})^2}$

$4 = \frac{25}{(2 + \frac{5}{2}k)^2}$

$(2 + \frac{5}{2}k)^2 = \frac{25}{4}$

$2 + \frac{5}{2}k = \frac{5}{2}$ (want $k > 0$)

$\frac{5k}{2} = \frac{5}{2} - 2 = \frac{1}{2}$ $\boxed{k = 1/5}$

$\Delta x = \frac{2(25)^{1/2}}{1/5} = 10 \cdot 5 = \boxed{50}$ ft!

units!

e) $kv_0^{1/2} = \frac{1}{5}(25)^{1/2} = \frac{1}{5}(5) = 1$

$x - x_0 = 50\left(1 - \frac{2}{2+t}\right) = 49$

$= 50\left(\frac{2+t-2}{2+t}\right)$

$= \frac{50t}{t+2} \longrightarrow$ $50t = 49(t+2)$

$= 49t + 98$

$(50-49)t = 98$

$\boxed{t = 98}$ min

units!

The whole point of this problem is to quantify how far a body coasts (until it comes to a stop).