MAT1505-03/04 19F Quiz 9 Print Name (Last, First) ____________, ____________

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation when appropriate). Indicate where technology is used and what type (Maple, GC).

1. The following Taylor series is easy to get by the geometric series trick and integration:

$$\arctan(x) = \sum_{n=0}^{\infty} \left((-1)^n \frac{x^{2n+1}}{2n+1} \right), |x| \le 1.$$

The reason is that its antiderivative $\frac{1}{x^2+1}$ is a function which can be manipulated from the geometric series identity and then integrated term by term to get the arctan Taylor series: $\arctan(x) = \int_0^x \frac{1}{x^2+1}\, dx$

a) Use the geometric series algebra manipulation trick to find this power series formula and write out the first 5 nonzero terms.

b) Why is its radius of convergence equal to 1?

c) Why does this series converge at the endpoints of the interval of convergence? Explain.

d) Use the Taylor series formula $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ to evaluate the cubic Taylor approximation for $\arctan(x)$ at the origin and compare with the expression from part a) for its first 2 nonzero terms. [Hint: Use Maple to evaluate the necessary derivatives at $x = 0$.]

▶ solution

a) $\arctan(x) = \int_0^x (1+x^2)^{-1}\, dx = \int_0^x \frac{1}{(1-(-x^2))}\, dx = \int_0^x \sum_{n=0}^{\infty} (-x^2)^n\, dx = \int_0^x \sum_{n=0}^{\infty} (-1)^n x^{2n}\, dx$

$= \sum_{n=0}^{\infty} (-1)^n \int_0^x x^{2n}\, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

$= \boxed{x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots}$

b) for $|-x^2| < 1 \rightarrow |x| < 1$

geometric series ratio for convergence

c) interval of convergence endpoints: $x = \pm 1$

$\sum_{n=0}^{\infty} (-1)^n \frac{(\pm 1)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n (\pm 1)}{2n+1} = \pm 1 \underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}}$

alternating series of terms decreasing to zero in absolute value, so must converge

[Remark. Though not requested, the corresponding absolute value series diverges at the endpoints, by the integral test: $\int_1^{\infty} \frac{dx}{2x+1} = \infty$]

d) $f(x) = \arctan x$

$f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1}$

$f''(x) = -(1+x^2)^{-2}(2x) = -\frac{2x}{(1+x^2)^2}$

$f'''(x) = -2\left[\frac{(1+x^2)^2 \cdot 1 - x \cdot 2(1+x^2)(2x)}{(1+x^2)^{\cancel{4}}}\right]$

$= -2\frac{(1+x^2)(1+x^2-4x^2)}{(1+x^2)^4}$

$= -\frac{2(1-3x^2)}{(1+x^2)^3}$

$f(0) = 0$

$f'(0) = 1$

$f''(0) = 0$

$f'''(0) = -2$

$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$

$= x - \frac{2}{6}x^3 + \ldots$

$= \underbrace{\boxed{x - \frac{1}{3}x^3}}_{T_3(x)} + \ldots$