MAT1505-03/04 19F Quiz 7   Print Name (Last, First) ________________, ________________

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation when appropriate). Indicate where technology is used and what type (Maple, GC).

From Maple's Help: The Maxwell distribution is a continuous probability distribution with probability density function (PDF!) given by:

$$f(x) = \begin{cases} 0 & x < 0 \\ \dfrac{\sqrt{2}\sqrt{\dfrac{1}{\pi}}\, x^2 e^{-\frac{x^2}{2\alpha^2}}}{\alpha^3} & \textit{otherwise} \end{cases}$$

a) Transform the probability integral $\int_0^\infty f(x)\,dx$ from the random variable x to the standard random variable $u = \frac{x}{\alpha}$. What is the new probability integral? Evaluate it in Maple to make sure you get the value 1 for total probability. Did you? **Set $\alpha = 1$ for the rest of this problem** which deals only with the "standard" distribution function.

b) Find the approximate (3 sig figs) expected value $\mu = \int_0^\infty x\, f(x)\,dx$ for the standard distribution using Maple.

c) Use calculus (derivative properties!) by hand to find the **exact** and approximate value (3 sig figs) x_p of the random variable where the peak of the standard distribution occurs.

d) Plot the standard distribution function for $x = 0..5$. Does part c) agree with your plot? Make a rough hand sketch on this sheet, locating the peak by a vertical line down from it to x_p on the axis and mark off the mean value by a vertical line up to the graph from μ on the axis.

e) What is the probability that the random variable x assumes a value less than the peak value? What is the equivalent percent to the nearest integer? Set up the integral and evaluate it using Maple.

f) **Optional. (Ignore till after the quiz).** Find the median value X (the value of the random variable for which the probability of having a value below it is equal to one half). Mark this off on your plot with a vertical line above this value. Does it seem reasonable? Why?

a) $u = \frac{x}{\alpha}$ $\quad du = \frac{dx}{\alpha}$ $\quad x=0:\ u=0$

$x = \alpha u$ $\quad dx = \alpha\, du$ $\quad x\to\infty:\ u\to\infty$

$$\int_0^\infty \sqrt{\frac{2}{\pi}}\,\underbrace{\frac{x^2}{\alpha^3}}_{\frac{(\alpha u)^2}{\alpha^3}}\,\underbrace{e^{-x^2/2\alpha^2}}_{e^{-u^2/2}}\,\underbrace{dx}_{\alpha\, du}$$

$$= \int_0^\infty \sqrt{\frac{2}{\pi}}\,\frac{\alpha^2}{\alpha^3}\,u^2 e^{-\frac{u^2}{2}}\,\alpha\, du$$

$$\boxed{= \int_0^\infty \sqrt{\frac{2}{\pi}}\,u^2 e^{-u^2/2}\,du \overset{\text{Maple}}{=} 1}$$

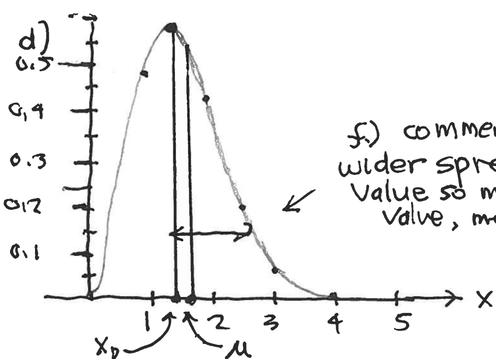
like setting $\alpha = 1$!

b) $\mu = \int_0^\infty \sqrt{\frac{2}{\pi}}\,x^3 e^{-x^2/2}\,dx \overset{\text{Maple}}{=} \boxed{2\sqrt{\frac{2}{\pi}}}$

$1.5958\ldots \to \boxed{\approx 1.60}$

c) $0 = \frac{d}{dx}\left(x^2 e^{-x^2/2}\right) = 2x e^{-x^2/2} + x^2 e^{-x^2/2}(-x)$

$= e^{-x^2/2}\,x\,(2-x^2) \to x^2 = 2;\ \boxed{x = \sqrt{2} = x_p \approx 1.41}$



f) comment: wider spread to right of peak value so more area right of peak value, median must be to right of peak

e) $P(0 \le x \le x_p) = \int_0^{\sqrt{2}} \sqrt{\frac{2}{\pi}}\,x^2 e^{-x^2/2}\,dx \overset{\text{Maple}}{\approx} \boxed{0.42759}$

$\approx \boxed{0.43}$

$\to \boxed{43\%}$

OPTIONAL:

f) $\frac{1}{2} = P(0 \le x \le X) = \int_0^X \sqrt{\frac{2}{\pi}}\,x^2 e^{-x^2/2}\,dx$

solve numerically Maple $\to$ $\boxed{X \approx 1.5382}$

so $x_p < X < \mu$