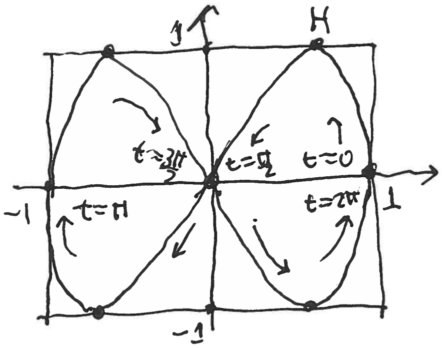
MAT1505-03/04 19F Final Exam Answers

(1) $x=\cos t,\ y=\sin 2t \quad 0\le t\le 2\pi$



$t=0..\pi/2$ is segment in first quadrant:

a) $A = -\int_0^{\frac{\pi}{2}} \underbrace{\underbrace{y(t)}_{>0}\ \underbrace{x'(t)}_{<0}}_{<0}\,dt > 0$

$= -\int_0^{\frac{\pi}{2}} (\sin 2t)(-\sin t)\,dt$

$= \int_0^{\pi/2} \sin t \sin 2t\,dt \overset{\text{Maple}}{=} \boxed{2/3 \approx 0.6667}$

Yes, fills about 2/3 of unit square containing this curve segment.

b) $x'=-\sin t \quad y'=2\cos 2t$

$m(t)=\frac{y'(t)}{x'(t)}=\frac{2\cos 2t}{-\sin t}=\boxed{-\frac{2\cos 2t}{\sin t}}$

origin: $t=\pi/2,\ 3\pi/2$

$m(\frac{\pi}{2}) = -\frac{2\cos\pi}{\sin\pi/2} = \frac{-2(-1)}{1} = \boxed{2}$

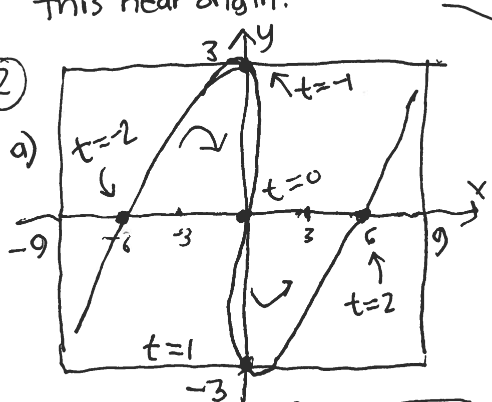
$m(\frac{3\pi}{2}) = -\frac{2\cos 3\pi}{\sin\frac{3\pi}{2}} = -\frac{2(-1)}{(-1)} = \boxed{-2}$

tan lines are $\boxed{y=\pm 2x}$

with gridlines gridbox count confirms this near origin!

(2)

a)



$\boxed{\text{traces out left to right}}$

(2) continued) $0=x=t^3-t=t(t^2-1)\to t=0,1,-1$

$0=y=t^3-4t=t(t^2-4)\to t=0,2,-2$

y intercepts: $x=0\to y=0,\ \underbrace{1^3-4(1)}_{-3},\ \underbrace{(-1)^3-4(-1)}_{3} \to \boxed{0,\pm 3}$

x intercepts: $y=0\to x=0,\ \underbrace{2^3-2}_{6},\ \underbrace{(-2)^3-(-2)}_{-6} \to \boxed{0,\pm 6}$

bulge into 1st quadrant: $t=-1..0$

b) At $t=2$, $x=6$ $(y=0)$.

$m(t)=\frac{y'(t)}{x'(t)}=\frac{3t^2-4}{3t^2-1} \qquad m(2)=\frac{3(4)-4}{3(4)-1}=\boxed{\frac{8}{11}\approx 0.73}$

looking at gridboxes can estimate $m\approx 3/4 = .75$ so good.

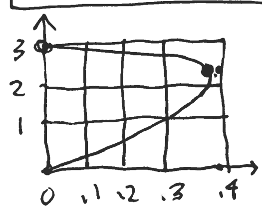
c) $m(t)\to\infty$ when $3t^2-1=0,\ t^2=1/3,\ t=\pm 1/\sqrt{3}$.

since $t<0$ for that part of curve segment, choose $t=-1/\sqrt{3}$

$x=\left(\frac{-1}{\sqrt{3}}\right)\left(\left(-\frac{1}{\sqrt{3}}\right)^2-1\right)=-\frac{1}{\sqrt{3}}\left(-\frac{2}{3}\right)=\frac{2}{3\sqrt{3}}$

$y=\left(\frac{-1}{\sqrt{3}}\right)\left(\left(\frac{-1}{\sqrt{3}}\right)^2-4\right)=\frac{-1}{\sqrt{3}}\left(\frac{1}{3}-4\right)=\frac{11}{3\sqrt{3}}$

$\boxed{(x,y)=\left(\frac{2}{3\sqrt{3}},\frac{11}{3\sqrt{3}}\right)\approx(.384, 2.12)}$



if plot $t=-1..0$ see this. gridlines confirm these values

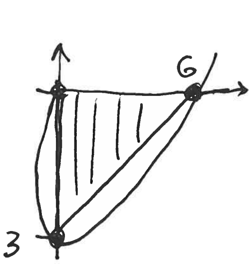
d) $x'=3t^2-1 \qquad x'^2=9t^4-6t^2+1$

$y'=3t^2-4 \qquad y'^2=9t^4-24t^2+16$

$x'^2+y'^2=18t^4-30t^2+17$

$L=\int_0^2\sqrt{x'^2+y'^2}\,dt$ ($t=0..2$ connects last 2 x-intercepts)

$=\boxed{\int_0^2\sqrt{18t^4-30t^2+17}\,dt} \overset{\text{Maple}}{=}$ yuck! $\approx 10.03428778 \approx \boxed{10.0329}$



side + hypotenuse $=3+3\sqrt{1^2+2^2}=3+3\sqrt{5}$

≈ 9.71 a bit shorter yes!

(1) c) $0=m(t)=-\frac{2\cos 2t}{\sin t}\to \cos 2t=0 \to 2t=\pm\frac{\pi}{2},\ t=\pm\frac{\pi}{4}$

first quadrant: $t=\pi/4\to (x,y)=(\cos\frac{\pi}{4},\sin\frac{\pi}{2})=(\frac{1}{\sqrt{2}},1)$

consistent with plot $\approx (0.71, 1)$

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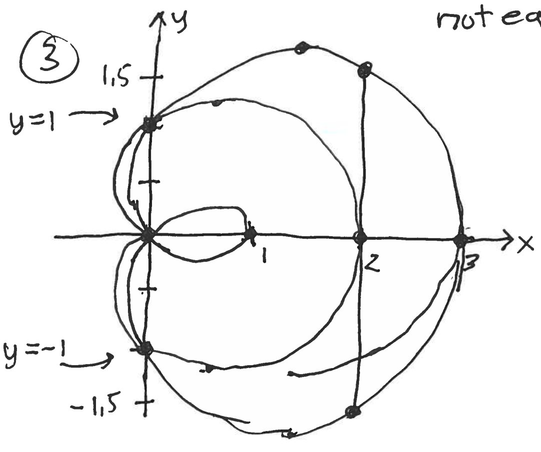
not easy to sketch!

(3)



a) $1+\cos\theta = 1+2\cos\theta$

$\cos\theta = 2\cos\theta \to \cos\theta = 0 \to r=1$

$\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$

$x = r\cos\theta = 0$

$y = r\sin\theta = 1\sin(\pm\frac{\pi}{2}) = \pm 1$

$\boxed{(x,y)} = \boxed{(0,\pm 1)}$ are the obvious intersection points

$\boxed{(r,\theta) = (1,\pm\frac{\pi}{2})}$ (or $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$)

b) need continuous angular interval for limits of integration so

$\theta = -\frac{\pi}{2} .. \frac{\pi}{2}$

$$A_1 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} r_2^2(\theta) - \frac{1}{2} r_1^2(\theta)\, d\theta \quad (r_2 > r_1)$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{1}{2}\left((1+2\cos\theta)^2 - (1+\cos\theta)^2\right) d\theta$$

$\cancel{1} + 4\cos\theta + 4\cos^2\theta$

$-\cancel{1} - 2\cos\theta - \cos^2\theta$

$2\cos\theta + 3\cos^2\theta$

$$= \int_0^{\frac{\pi}{2}} 2\cos\theta + 3\cos^2\theta\, d\theta \overset{\text{Maple}}{=} \boxed{\frac{3\pi}{4} + 2} \approx \boxed{4.3562}$$

e) positive y-intercept at $y=1$, $x=0$: $\theta = \frac{\pi}{2}$

$x(\theta) = r(\theta)\cos\theta = (1+\cos\theta)\cos\theta, (1+2\cos\theta)\cos\theta$

$y(\theta) = r(\theta)\sin\theta = (1+\cos\theta)\sin\theta, (1+2\cos\theta)\sin\theta$

$x'(\theta) = \frac{d}{d\theta}(\cos\theta + \cos^2\theta)$ $\qquad$ $\frac{d}{d\theta}(\cos\theta + 2\cos^2\theta)$

$= -\sin\theta - 2\cos\theta\sin\theta$ $\qquad$ $= -\sin\theta - 4\cos\theta\sin\theta$

$= -\sin\theta(1+2\cos\theta)$ $\qquad$ $= -\sin\theta(1+4\cos\theta)$

$y'(\theta) = \frac{d}{d\theta}(\sin\theta + \cos\theta\sin\theta)$ $\qquad$ $\frac{d}{d\theta}(\sin\theta + 2\cos\theta\sin\theta)$

$= \cos\theta - \sin^2\theta + \cos^2\theta$ $\qquad$ $= \cos\theta - 2\sin^2\theta + 2\cos^2\theta$

$$\frac{dy}{dx}(\theta) = \frac{y'(\theta)}{x'(\theta)} = \frac{\cos\theta + \cos^2\theta - \sin^2\theta}{-\sin\theta(1+2\cos\theta)}, \quad \frac{\cos\theta + 2\cos^2\theta - 2\sin^2\theta}{-\sin\theta(1+4\cos\theta)}$$

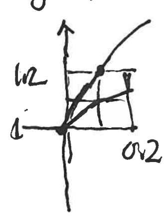
$$\frac{dy}{dx}\left(\frac{\pi}{2}\right) = \frac{-1}{-1} = 1, \quad \frac{-2}{-1} = 2$$

$1 = m_1 = \tan\phi_1$ $\qquad$ $2 = m_2 = \tan\phi_2$

$\boxed{\phi_1 = \arctan 1} = \frac{\pi}{4} = 45°$ $\qquad$ $\boxed{\phi_2 = \arctan 2} \approx 63.4$

$\phi_2 - \phi_1 \approx 63.4° - 45° \approx \boxed{18.4°}$

$y = 1 + m_i x = 1+x, 1+2x$



zoom in, grid boxes agree with slopes about 1 and 2