MAT1505-03/04 19F Takehome test 3 Answers

(1) $e^{+x} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ 2 ways to get Taylor series

a) $e^x = e^{-2+(x+2)} = e^{-2}e^{x+2} = e^{-2}\sum_{n=0}^{\infty}\frac{(x+2)^n}{n!}$

$= e^{-2}\left(1 + (x+2) + \frac{(x+2)^2}{2!} + \frac{(x+2)^3}{3!} + \dots\right)$

b) $f(x) = e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(-2)(x+2)^n}{n!}$

$f^{(n)}(x) = e^x$

$f^{(n)}(-2) = e^{-2}$

$x-(-2) = x+2$

$= \sum_{n=0}^{\infty} \frac{e^{-2}(x+2)^n}{n!}$

$e^{-2.1} = \sum_{n=0}^{\infty}\frac{e^{-2}(-2.1+2)^n}{n!} = \boxed{\sum_{n=0}^{\infty}\frac{e^{-2}(-0.1)^n}{n!}}$

$= \boxed{\sum_{n=0}^{\infty}(-1)^n\frac{e^{-2}(0.1)^n}{n!}}$

$= \underbrace{e^{-2} - (0.1)e^{-2} + \underbrace{\frac{(0.1)^2}{2}e^{-2}}_{\approx .000677}}_{} - \underbrace{\frac{(0.1)^3}{6}e^{-2}}_{\approx 0.0000226 < .0005} + \frac{(0.1)^4}{24}e^{-2} - \dots$

$T_2(-2.1) \approx 0.122478$

So to 3 decimal places, the quadratic approximation (first 3 terms) gives $\boxed{e^{-2.1} \approx 0.122}$

Maple gives $e^{-2.1} \approx 0.1224564$

which rounds to the same 3 decimal places

For the series expansion at $x=0$:

$e^{-2.1} = \sum_{n=0}^{\infty}\underbrace{\frac{(-2.1)^n}{n!}}_{a_n}$ $\quad a_{10} = 0.00045965$, $|a_9| = 0.00219$

The 10th power (eleventh) term goes under 0.0005

so $\boxed{\text{ten terms}}$ would be needed

(2) a) $N(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} = \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\frac{(-x^2/2)^n}{n!}$

$= \boxed{\frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{2^n n!}}$

b) $P = 2\int_0^1 N(x)\,dx = 2\int_0^1\frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^n n!}dx$

$= \frac{2}{\sqrt{2\pi}}\sum_{n=0}^{\infty}(-1)^n\frac{\int_0^1 x^{2n}dx}{2^n n!} = \frac{2}{\sqrt{2\pi}}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2^n(2n+1)n!}\Big|_0^1$

$= \frac{2}{\sqrt{2\pi}}\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n(2n+1)n!} = \sum_{n=0}^{\infty}(-1)^n\underbrace{\frac{2}{\sqrt{2\pi}\,2^n(2n+1)n!}}_{a_n}$

(2) b) continued: alternating series estimate

$a_3 \approx 0.00237$

$a_4 \approx 0.00023 > .00005$

$a_5 \approx 0.0000189 < .00005$

so only the $\boxed{\text{first 5 terms}}$ $(a_0, \dots, a_4)$ are needed giving

$P \approx 0.682707 \approx \boxed{0.6827 \text{ to 4 decimal places}}$

Maple gives $P \approx 0.6826894920$ "exactly" ≈ 0.6827 rounding to 4 decimal places

$\boxed{\text{so Maple agrees!}}$

(2) c) see next page

(3) $a_n = \frac{(2x-1)^{2n+1}}{3^{3n}(n+1)^{3/2}} \to \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$

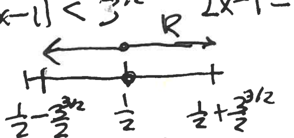
$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\left|\frac{(2x-1)^{2(n+1)+1}}{3^{3(n+1)}(n+1+1)^{3/2}}\right|}{\left|\frac{(2x-1)^{2n+1}}{3^{3n}(n+1)^{3/2}}\right|}$

$= \frac{|2x-1|^{(2n+3)-(2n+1)}}{3^{3n+3-3n}}\cdot\left(\frac{n+1}{n+2}\right)^{3/2}$

$= \underbrace{\left(\frac{n+1}{n+2}\right)^{3/2}}_{\to 1 \text{ as } n\to\infty}\frac{|2x-1|^2}{3^3} \xrightarrow[n\to\infty]{} \frac{|2x-1|^2}{3^3} < 1$

$|2x-1|^2 < 3^3$ $\quad |2x-1| < 3^{3/2}$ $\quad 2x-1 = \pm 3^{3/2}$

$|x-\frac{1}{2}| < \boxed{\frac{3^{3/2}}{2} = R}$ radius of convergence



$[\frac{1}{2}-\frac{1}{2}3^{3/2}, \frac{1}{2}+\frac{1}{2}3^{3/2}]$

endpoints $(2x-1) = \pm 3^{3/2}$

$\sum_{n=0}^{\infty}\frac{(\pm 3^{3/2})^{2n+1}}{3^{3n}(n+1)^{3/2}} = \sum_{n=0}^{\infty}\frac{\pm 3^{3n}3^{3/2}}{3^{3n}(n+1)^{3/2}}$

$= \pm\sum_{n=0}^{\infty}\frac{3^{3/2}}{(n+1)^{3/2}}$ $\underset{\text{large } n}{\sim}$ $p = 3/2 > 1$ series so converges at both endpoints

interval of convergence:	endpt values:
$[\frac{1}{2}-\frac{1}{2}3^{3/2}, \frac{1}{2}+\frac{1}{2}3^{3/2}]$ $\frac{1}{2}-\frac{1}{2}3^{3/2} \le x \le \frac{1}{2}+\frac{1}{2}3^{3/2}$ radius of convergence: $\frac{1}{2}3^{3/2}$	$\pm 3^{3/2}\zeta(3/2)$ ± 13.5743

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④ a) $f(r) \approx \frac{\ln(2)}{0.01r}\left(1-\frac{0.01r}{2}\right)^{-1}$ $\qquad (1+x)^{-1} \approx \underbrace{1+(-1)x}_{\text{first two terms}} + \frac{(-1)(-2)}{2}x^2+\ldots$

$\approx 1+(-1)\left(-\frac{0.01r}{2}\right) = 1+.005r$

$f(r) \approx \left(\frac{100\ln(2)}{r}\right)(1+\underbrace{.005r}_{R=.005r})$

b) $f(9) \approx 100(\ln 2)\ 1.045 \approx 72.43 \leftrightarrow f(9) = 72.780$

$f(2) \approx 100(\ln 2)\ 1.01 \approx 70.01 \leftrightarrow f(2) = 70.008$

exact (numerical): pretty close

⑤ a) $E = \frac{q}{R^2} - \frac{q}{(R+d)^2} = \frac{q}{R^2} - \frac{q}{(R(1+d/R))^2} = \frac{q}{R^2} - \frac{q}{R^2(1+d/R)^2}$

$= \frac{q}{R^2}\left(1-(1+d/R)^{-2}\right) = \frac{q}{R^2}\left(\cancel{1}-\left(\cancel{1}+(-2)(d/R)+\frac{(-2)(-3)}{1\cdot 2}(d/R)^2+\ldots\right)\right)$

$= \frac{q}{R^2}\left(\frac{2d}{R}+\ldots\right) \approx \underbrace{\frac{q}{R^2}}_{E_0}\left(\frac{2d}{R}\right)$

$\approx \boxed{\frac{2qd}{R^3}}$ to leading order $\qquad \boxed{\frac{E}{E_0} \approx \frac{2d}{R}}$

length dimensions cancel
so dimensionless
i.e. both E, E_0 have
same dimensions

② c) $N(x) = \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^n n!} = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{2^0 0!}-\frac{x^2}{2^1 1!}+\ldots\right) = \frac{1}{\sqrt{2\pi}}\left(1-\frac{1}{2}x^2+\ldots\right)$

$N(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$

$N'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\left(-\frac{2x}{2}\right) = \frac{-xe^{-x^2/2}}{\sqrt{2\pi}}$

$N''(x) = -\frac{1}{\sqrt{2\pi}}\left(1e^{-x^2/2}+xe^{-x^2/2}\left(-\frac{2x}{2}\right)\right) = -\frac{1}{\sqrt{2\pi}}(1-x^2)$

$N(0) = \frac{1}{\sqrt{2\pi}}, \quad N'(0) = 0, \quad N''(0) = -\frac{1}{\sqrt{2\pi}}$

$N(x) = \sum_{n=0}^{\infty}\frac{N^{(n)}(0)x^n}{n!} = N(0)+N'(0)x+\frac{N''(0)x^2}{2}+\ldots$

$= \frac{1}{\sqrt{2\pi}}\left(1-\frac{1}{2}x^2+\ldots\right)$ agrees with above formula.