Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT **(but give decimal approximations for interpretation)**. Indicate where technology is used and what type (Maple, GC).

1. $9y''(t) + 6y'(t) + 10y(t) = 37\cos(t)$, $y(0)=0$, $y'(0)=0$ [Maple notation].
a) State Maple's solution of the initial value problem (use function notation $y(t)$).
b) Put the DE into standard linear form (unit leading coefficient). Then identify the values of the damping constant and characteristic time $k_0 = 1/\tau_0$, the natural frequency ω_0, and the quality factor $Q = \omega_0\tau_0$, exactly and numerically. Is this underdamped, critically damped or overdamped?
c) Find the general solution by hand, showing all steps.
d) Find the solution satisfying the initial conditions, showing all steps.
e) Give exact and numerical values of the amplitude and phase shift of the steady state solution (particular solution!) and re-express the sinusoidal factor of this solution in phase-shifted cosine form. [Make sure you use a diagram to justify your values.] State what numerical fraction of a cycle (2π) the phase shift is (i.e., evaluate $\delta/2\pi$) as well as its numerical value in degrees, and whether the cosine curve is shifted left (earlier in time) or right (later in time) on the time line (by a phase less than or equal to half a cycle of course). Explain.
f) Make a rough sketch of the plot of your full solution with the steady state solution until they merge.
g) **Optional.** Find the two envelope functions of the decaying oscillating transient solution.

► solution

1) a) $y(t) = \underbrace{e^{-t/3}(-\cos t - \frac{19}{3}\sin t)}_{\text{transient}} + \underbrace{\cos t + 6\sin t}_{\text{steady state}}$

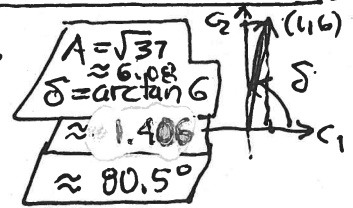b) $y'' + \frac{2}{3}y' + \frac{10}{9}y = \frac{37}{9}\cos t$

$\boxed{k_0 = \frac{2}{3}}$ ω_0^2

$C_0 = \frac{3}{2} = 1.5$ $\omega_0 = \frac{\sqrt{10}}{3} \approx 1.054$

$Q = \omega_0\tau_0 = \frac{\sqrt{10}}{2} \approx 1.58 > 0.5$ underdamped

c) $y_n = e^{rx} \to 9r^2 + 6r + 10 = 0 \to r = -\frac{1}{3} \pm i$

$e^{rx} = e^{-t/3}e^{\pm it} = e^{-t/3}(\cos t \pm i\sin t)$

$\hookrightarrow e^{-t/3}\cos t, \ e^{-t/3}\sin t$ real basis

$y_h = e^{-t/3}(C_1\cos t + C_2\sin t)$

$10[y_p = C_3\cos t + C_4\sin t]$
$6[y_p' = -C_3\sin t + C_4\cos t]$
$9[y_p'' = -C_3\cos t - C_4\sin t]$

note: $r = 3$
$5\tau = 15$

$9y_p'' + 6y_p' + 10y_p = [(10-9)C_3 + 6C_4]\cos t + [-6C_3 + (10-9)C_4]\sin t = 37\cos t$
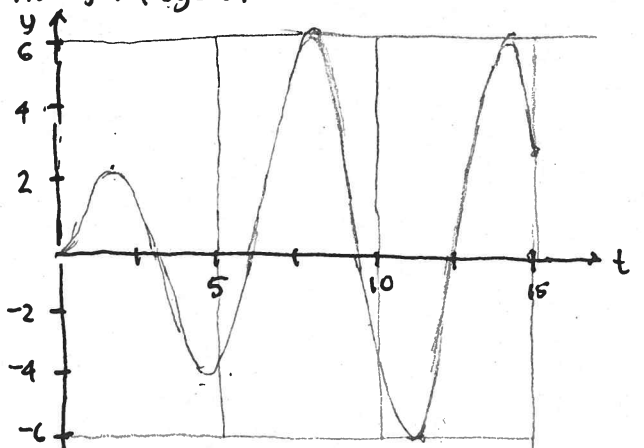
$C_3 + 6C_4 = 37$
$-6C_3 + C_4 = 0$

$\begin{bmatrix} 1 & 6 \\ -6 & 1 \end{bmatrix}\begin{bmatrix} C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 37 \\ 0 \end{bmatrix}$

$\begin{bmatrix} C_3 \\ C_4 \end{bmatrix} = \frac{1}{36+1}\begin{bmatrix} 1 & -6 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} 37 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$
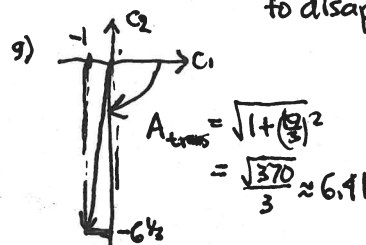
$\underline{y_p = \cos t + 6\sin t}$

$\boxed{y = e^{-t/3}(C_1\cos t + C_2\sin t) + \cos t + 6\sin t}$ gen soln

) $y' = e^{-t/3}(-C_1\sin t + C_2\cos t) + -\sin t + 6\cos t$
$\quad -\frac{1}{3}e^{-t/3}(C_1\cos t + C_2\sin t)$

d) continued:
$y(0) = C_1 + 1 = 0 \to C_1 = -1$
$y'(0) = C_2 + 6 -\frac{1}{3}C_1 = 0 \to C_2 = \frac{1}{3}(-1) - 6 = -\frac{19}{3}$

$\boxed{y = e^{-t/3}(-\cos t - \frac{19}{3}\sin t) + \cos t + 6\sin t}$ IVP soln

e) $\boxed{y_{ss} = \cos t + 6\sin t}$:
$= \sqrt{37}\cos(t - \arctan 6)$

$\boxed{\frac{\delta}{2\pi} \approx .224 \text{ cycle}}$

$A = \sqrt{37} \approx 6.08$
$\delta = \arctan 6 \approx 1.406$
$\approx 80.5°$

cosine curve shifted right (later in time) by nearly ¼ cycle.

f)

decay window $t = 0 \sim 5\tau$ for transient to disappear

g) $A_{trans} = \sqrt{1 + (\frac{19}{3})^2}$
$= \frac{\sqrt{370}}{3} \approx 6.41$

$y_{env} = \pm\frac{\sqrt{370}}{3}e^{-t/3}$