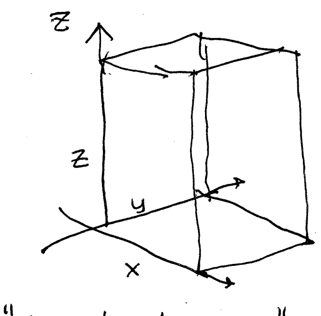
MAT2500-01/04 17S Quiz 7   Print Name (Last, First) ____________________________|__

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation when appropriate). Indicate where technology is used and what type (Maple, GC).

A package in the shape of a rectangular box can be mailed through the US Postal Service if the sum of its length and girth (girth equals the perimeter of the rectangular cross-section perpendicular to the longest dimension "length") is at most 108 in. Find the dimensions and volume (in cubic ft) of the package with the largest volume that can be mailed. Verify the local max with the second derivative test, clearly explained. Be sure to indicate what your variable names stand for and make a diagram of the region of the plane where you consider this 2d max/min problem, indicating the conditions which lead to its boundaries.

▶ solution



"box standing up"

x, y lateral dimensions

z length

girth: $2(x+y)$

$S = 2(x+y) + z$ "length plus girth"

largest box will have $S = 108$ so $\boxed{2x+2y+z = 108}$ (constraint)

$\hookrightarrow z = 108 - 2x - 2y$ (use to eliminate one variable)

$V = xyz = xy(108-2x-2y)$

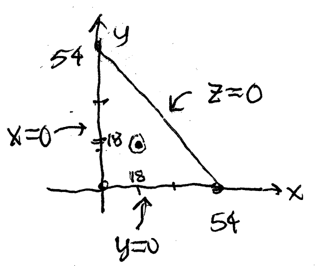
$= 108xy - 2x^2y - 2xy^2$

so maximize $\boxed{V = 108xy - 2x^2y - 2xy^2}$

$$0 = \frac{\partial V}{\partial x} = \frac{\partial}{\partial x}(108xy - 2x^2y - 2xy^2) = 108y - 4xy - 2y^2 = 2y(54 - 2x - y)$$

$$0 = \frac{\partial V}{\partial y} = \frac{\partial}{\partial y}(108xy - 2x^2y - 2xy^2) = 108x - 2x^2 - 4xy = 2x(54 - x - 2y)$$

but $x > 0$, $y > 0$, $z = 108 - 2x - 2y > 0$ or $y + x - 54 < 0$ $\quad y < 54 - x$



$V = 0$ on triangular region boundary but $V > 0$ inside so must have a maximum inside (allowed region)

$2[2x + y = 54) \to 4x + 2y = 108$

$x + 2y = 54 \to \quad x + 2y = 54$

$3x = 54 \qquad x = 54/3 = 18$

$y = 54 - 2x = 54 - 36 = 18$

$z = 108 - 2(18) - 2(18) = 36$

must be global maximum pt $\quad (x, y, z) = (18, 18, 36)$

$V = 18 \cdot 18 \cdot 36 = 11664 \text{ in}^3 \frac{\text{ft}^3}{(12\text{ in})^3}$

$= (1.5)(1.5)(3) \text{ ft}^3 = 6.75 \text{ ft}^3$

$\boxed{\text{lateral sides of 18 inches and length of 36 in yields the biggest volume of } 6.75 \text{ ft}^3}$

2nd derivative check

$\frac{\partial^2 V}{\partial x^2} = \frac{\partial}{\partial x}(108y - 4xy - 2y^2) = -4y \qquad \left.\frac{\partial^2 V}{\partial x^2}\right|_{\substack{x=18\\y=18}} = -4(18) < 0$ local max's

$\frac{\partial^2 V}{\partial y^2} = \frac{\partial}{\partial y}(108x - 2x^2 - 4xy) = -4x \qquad \left.\frac{\partial^2 V}{\partial y^2}\right|_{\substack{x=18\\y=18}} = -4(18) < 0$

$\frac{\partial^2 V}{\partial x \partial y} = \frac{\partial}{\partial y}(108y - 4xy - 2y^2) = 108 - 4x - 4y \qquad \left.\frac{\partial^2 V}{\partial x \partial y}\right|_{\substack{x=18\\y=18}} = 108 - 8(18) = -36$

$\frac{\partial^2 V}{\partial x^2}\frac{\partial^2 V}{\partial y^2} - \left(\frac{\partial^2 V}{\partial x \partial y}\right)^2 = (72)^2 - (-36)^2 > 0$ so is local max in all directions.

A word problem gets a word answer independent of whatever variable names were introduced to solve it.

MAT2500-01/04 17S Quiz 7 Alternate soln

$2x+2y+z = 108$

x, y on equal footing
expect $x=y$ for soln

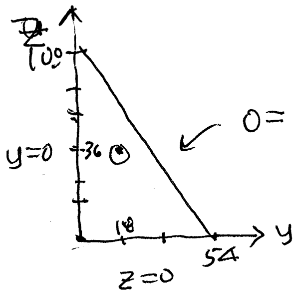
"other" variable so natural to eliminate using this constraint as done in solution on reverse page

BUT one could choose to eliminate x or y.

say: $x = \dfrac{108-2y-z}{2} = 54 - y - \tfrac{1}{2}z \quad \begin{matrix} > 0 \\ = 0 \end{matrix} \rightarrow y = 54 - \tfrac{1}{2}z$ or $z = 108-2y$

or

$0 = x = 54 - y - \tfrac{1}{2}z$



$$V = (54-y-\tfrac{1}{2}z)yz = 54yz - y^2z - \tfrac{1}{2}yz^2$$

$$\frac{\partial V}{\partial y} = 54z - 2yz - \tfrac{1}{2}z^2 = \tfrac{1}{2}z\,(\underbrace{108-4y-z}_{=0}) \rightarrow z = -4y+108$$

$$\frac{\partial V}{\partial z} = 54y - y^2 - yz = y(\underbrace{54-y-z}_{=0})$$

$54 - y = -4y + 108$

$3y = 54$

$y = \underline{18}$

$z = 108 - 4(18) = \underline{36}$

$x = 54 - 18 - \tfrac{1}{2}(36) = \underline{18}.$

etc.