MAT2500-01/04 17S Final Exam Answers (1)

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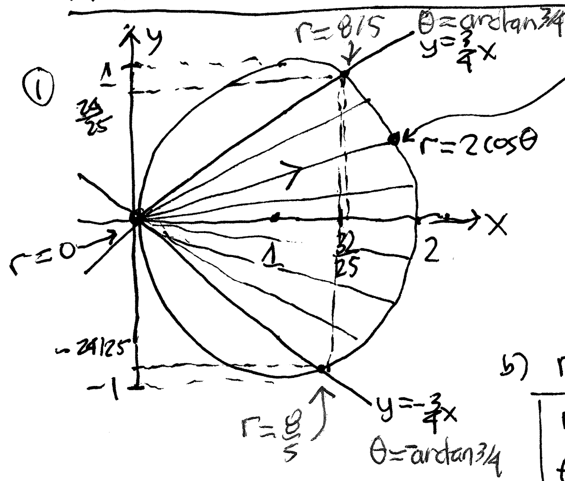


$(x-1)^2+y^2=1 \rightarrow C(1,0),\ r=1$ circle

$y^2=\frac{9}{16}x^2 \rightarrow y=\pm\frac{3}{4}x$ straight lines

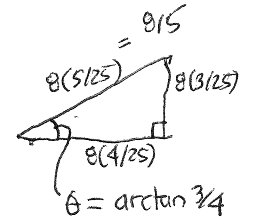
$x^2-2x+\cancel{1}+y^2=\cancel{1}$, $x^2+y^2=2x$

intersection: $x^2+\frac{9}{16}x^2=2x$

$\frac{25}{16}x^2-2x=0$

$\left(\frac{25x}{16}-2\right)x=0 \rightarrow x=0, \frac{32}{25}$

$y=0, \pm\frac{3}{4}\left(\frac{32}{25}\right)=\pm\frac{24}{25}$



$\boxed{(0,0),\ \left(\frac{32}{25},\frac{24}{25}\right),\ \left(\frac{32}{25},-\frac{24}{25}\right)}$

$r=0$; $r=8/5$, $\theta=\pm\arctan 3/4$

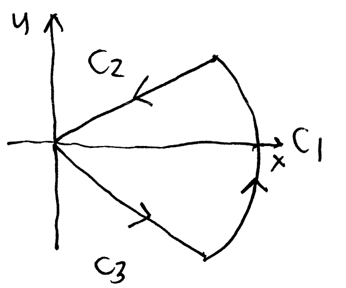
b) $r^2=2r\cos\theta$ $\quad\tan\theta=\pm\frac{3}{4}$

$\boxed{\begin{array}{l} r=2\cos\theta \\ \theta=\pm\arctan\frac{3}{4}\end{array}}$

c) $\vec{F}=\langle x-y, x\rangle$

$\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}=\frac{\partial}{\partial x}(x)-\frac{\partial}{\partial y}(x-y)=1-(-1)=2$

$$\iint_R \frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\,dA=\boxed{\int_{-\arctan 3/4}^{\arctan 3/4}\int_0^{2\cos\theta} 2\ \underbrace{r\,dr\,d\theta}_{dA}} \overset{\text{Maple}}{=} \boxed{\frac{48}{25}+4\arctan\left(\frac{3}{4}\right)} \approx 4.4940$$



$$\int_C \vec{F}\cdot d\vec{r}=\int_{C_1}F\cdot dr+\int_{C_2}F\cdot dr+\int_{C_3}\vec{F}\cdot dr$$

C_1: $x^2-2x+y^2=0 \rightarrow x=\frac{2\pm\sqrt{4-4y^2}}{2}=1\pm\sqrt{1-y^2} \rightarrow +$ sign.

$y=t=-\frac{24}{25}..\frac{24}{25}$ $\quad x=1-\sqrt{1-t^2}$

$\vec{r}(t)=\langle 1+\sqrt{1-t^2}, t\rangle$ $\quad \vec{r}'(t)=\left\langle \frac{1}{2}\frac{(-2t)}{\sqrt{1-t^2}}, 1\right\rangle=\left\langle\frac{-t}{\sqrt{1-t^2}}, 1\right\rangle$

$\vec{F}(\vec{r}(t))=\langle 1+\sqrt{1-t^2}-t, 1+\sqrt{1-t^2}\rangle$

$\vec{F}(\vec{r}(t))\cdot\vec{r}'(t)=\left(\frac{(1-t+\sqrt{1-t^2})(-t)}{\sqrt{1-t^2}}+1+\sqrt{1-t^2}\right)$

$\frac{t(t-1)}{\sqrt{1-t^2}}-t+1+\sqrt{1-t^2}$

$$\int_{C_1}\vec{F}(\vec{r}(t))\cdot\vec{r}'(t)\,dt=\int_{-\frac{24}{25}}^{\frac{24}{25}}\frac{t(t-1)}{\sqrt{1-t^2}}-t+1+\sqrt{1-t^2}\,dt \overset{\text{Maple}}{=} \boxed{\frac{48}{25}+2\arcsin\left(\frac{24}{25}\right)} \approx 4.4940$$

$\boxed{=\int_C \vec{F}\cdot d\vec{r}}$

see last page for check

C_2: $x=t$, $y=\frac{3}{4}t$ $\quad t=\frac{32}{25}..0$ (backwards)

$\vec{r}(t)=\langle t,\frac{3}{4}t\rangle$ $\quad \vec{r}'(t)=\langle 1, 3/4\rangle$ $\quad \vec{F}(\vec{r}(t))=\langle t(1-\frac{3}{4}), t\rangle=\langle\frac{t}{4}, t\rangle$

$\vec{F}(r(t))\cdot\vec{r}'(t)=\langle\frac{t}{4},t\rangle\cdot\langle 1, 3/4\rangle=\frac{t}{4}+\frac{3}{4}t=t$

$$\int_{C_2}\vec{F}\cdot d\vec{r}=\int_{\frac{32}{25}}^{0}t\,dt=\left.\frac{t^2}{2}\right|_{\frac{32}{25}}^{0}=-\frac{1}{2}\left(\frac{32}{25}\right)^2=-\frac{512}{625}$$

cancel

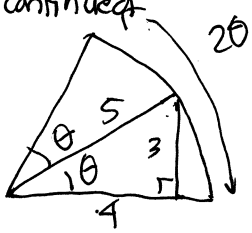
C_3: $x=t$, $y=-\frac{3}{4}t$, $t=0..\frac{32}{25}$ forwards.

$\vec{r}(t)=\langle t,-\frac{3}{4}t\rangle$ $\quad \vec{r}'(t)=\langle 1,-3/4\rangle$ $\quad F(\vec{r}(t))=\langle t(1+\frac{3}{4}), t\rangle$

$F(r(t))\cdot\vec{r}'(t)=\langle\frac{7}{4}t, t\rangle\cdot\langle 1,-\frac{3}{4}\rangle=\left(\frac{7}{4}-\frac{3}{4}\right)t=t.$

$$\int_{C_3}\vec{F}\cdot d\vec{r}=\int_0^{\frac{32}{25}}t\,dt=\left.\frac{t^2}{2}\right|_0^{\frac{32}{25}}=\frac{1}{2}\left(\frac{32}{25}\right)^2=\frac{512}{625}$$

(1) continued



$\theta = \arctan 3/4$

$\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right) = \frac{24}{25}$

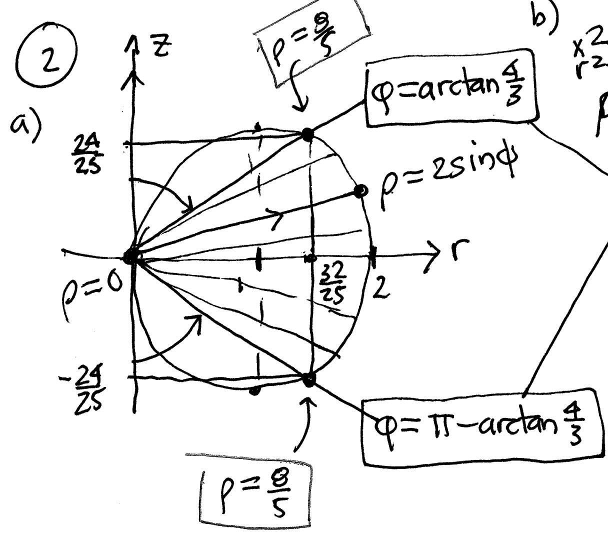
$2\theta = \arcsin\frac{24}{25}$

$4\theta = \boxed{4\arctan 3/4 = 2\arcsin\frac{24}{25}}$

so they agree exactly!

(2)

a)



b)

$x^2+y^2=2x$

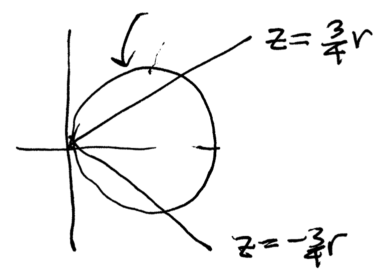
$r^2+z^2=2r$

$\rho^2 = 2r = 2\rho\sin\phi$

$\boxed{\rho = 2\sin\phi}$

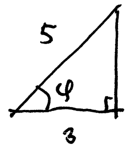
3 bounding curves.

$(r-1)^2+z^2=1$



intersection pts: $\rho = 2\sin\left(\arctan\frac{4}{3}\right)$

$= 2\left(\frac{4}{5}\right) = \frac{8}{5}$



c) $$\int_0^{2\pi}\int_{\pi-\arctan 4/3}^{\arctan 4/3}\int_0^{2\sin\phi} 1\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \overset{\text{Maple}}{=} \boxed{2\pi\left(\frac{856}{625} - 2\arctan\frac{4}{3} + \pi\right)}$$

≈ 16.6919

(3) a) $\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \frac{\partial}{\partial x}(36x+9y+6z) + \frac{\partial}{\partial y}(9x+9y+3z) + \frac{\partial}{\partial z}(6x+3y+4z) = 36 + 9 + 4 = \boxed{49}$

b) $\operatorname{curl}\vec{F} = \left\langle \frac{\partial}{\partial y}(6x+3y+4z) - \frac{\partial}{\partial z}(9x+9y+3z), \frac{\partial}{\partial z}(36x+9y+6z) - \frac{\partial}{\partial x}(6x+3y+4z), \frac{\partial}{\partial x}(9x+9y+3z) - \frac{\partial}{\partial y}(36x+9y+6z) \right\rangle$

$= \langle 3-3, 6-6, 9-9\rangle = \langle 0,0,0\rangle$

c) $\vec{\nabla} f(x,y,z) = \left\langle \frac{\partial}{\partial x}\left(18x^2+\frac{9}{2}y^2+2z^2+9xy+3yz+6xz\right), \frac{\partial}{\partial y}(\ldots), \frac{\partial}{\partial z}(\ldots)\right\rangle$

$= \langle 36x+9y+6z,\ 9y+9x+3z,\ 4z+3y+6x\rangle = \langle F_1, F_2, F_3\rangle$

d) $\vec{r}(t) = \langle \sin t, \sin t, \cos t\rangle \quad \vec{r}\,'(t) = \langle \cos t, \cos t, -\sin t\rangle$

$\vec{F}(\vec{r}(t)) = \ldots$ no! too much work, use potential function ⟶

But bob wants the line integral!

e) $\int_C \vec{F}\cdot d\vec{r} = f\left(\vec{r}\left(\frac{\pi}{2}\right)\right) - f(\vec{r}(0)) = f(0,0,1) - f(1,1,0)$

$= 2 - \left(18 + \frac{9}{2} + 9\right) \overset{\text{Maple}}{=} \boxed{\frac{59}{2}}$

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(3) d) $\vec F(\vec r(t)) = \langle 36\sin t + 9\sin t + 6\cos t,\ 9\sin t + 9\sin t + 3\cos t,\ 4\cos t + 3\sin t + 6\sin t\rangle$

$= \langle 45\sin t + 6\cos t,\ 18\sin t + 3\cos t,\ 9\sin t + 4\cos t\rangle$

$\cos t \qquad \cos t \qquad -\sin t$

$\vec F(\vec r(t))\cdot\vec r'(t) = (45\sin t + 6\cos t)\cos t + (18\sin t + 3\cos t)\cos t - (9\sin t + 4\cos t)\sin t$

$= (45+18-4)\sin t\cos t + (6+3)\cos^2 t - 9\sin^2 t.$

$\int_C \vec F\cdot d\vec r = \int_0^{\pi/2} (59\ \underbrace{\sin t}_{u}\ \underbrace{\cos t}_{du} + 9\cos^2 t - 9\sin^2 t)\,dt$

$\frac{u^2}{2} = \frac{\sin^2 t}{2}$

$= \frac{59}{2}\sin^2 t + \frac{9}{2}(\cos t\sin t + t) - \frac{9}{2}(-\cos t\sin t + t)\Big|_0^{\pi/2}$

$= \boxed{\frac{59}{2}} + \cancel{\left(\frac{9}{2} - \frac{3}{2}\right)\frac{\pi}{2} = \frac{59}{2}}$ ✓

(1) c) check with polar angle parametrization $r = 2\cos t,\ \theta = t$

$\vec r(t) = \langle (2\cos t)\cos t, (2\cos t)\sin t\rangle = \langle 2\cos^2 t, 2\cos t\sin t\rangle$

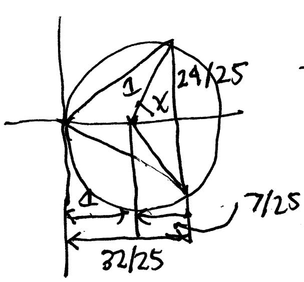
$\vec r'(t) = \langle -4\cos t\sin t, -2\sin^2 t + 2\cos^2 t\rangle$

$\vec F(\vec r(t)) = \langle x(t)-y(t), x(t)\rangle = \cancel{\langle -4\cos t\sin t + 2\sin^2 t - 2\cos^2 t,}$

$= \langle 2\cos^2 t - 2\cos t\sin t,\ 2\cos^2 t\rangle$

$\vec F(\vec r(t))\cdot\vec r'(t) = (2\cos^2 t - 2\cos t\sin t)(-4\cos t\sin t) + (-2\sin^2 t + 2\cos^2 t)(2\cos^2 t)$

$\int_{C_1}\vec F\cdot d\vec r = \int_{-\arctan 3/4}^{\arctan 3/4} \%\ dt \overset{\text{Maple}}{=} \frac{48}{25} + 4\arctan\frac{3}{4}$



$\tan x = \frac{24/25}{7/25} = \frac{24}{7},\ x = \arctan 24/7$

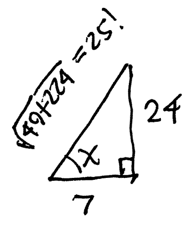
$\vec r(t) = \langle 1+\cos t, \sin t\rangle,\ \vec r'(t) = \langle -\sin t, \cos t\rangle$

$\vec F(\vec r(t)) = \langle 1+\cos t - \sin t,\ 1+\cos t\rangle$

$\vec F(\vec r(t))\cdot\vec r'(t) = -(1+\cos t - \sin t)\sin t + (1+\cos t)\cos t$

$\int_{C_1}\vec F\cdot d\vec r = \int_{-\arctan 24/7}^{\arctan 24/7} \%\ dt \overset{\text{Maple}}{=} \frac{48}{25} + 2\arctan\frac{24}{7}$

$= \frac{48}{25} + 2\arcsin\frac{24}{25}$!



$\sin x = \frac{24}{25} \to x = \arcsin\frac{24}{25}$

another Pythagorean triple!