MAT1505-03/04 17F Quiz 9 Print Name (Last, First) ________________, ________|____

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation when appropriate). Indicate where technology is used and what type (Maple, GC).

1.a) Use the geometric series algebra manipulation trick and the identity $\ln\left(\frac{2+x}{2-x}\right)+C=\int\frac{4}{4-x^2}\,dx$ to find a power series for $f(x)=\ln\left(\frac{2+x}{2-x}\right)$. [Hint: Set $C=0$ for the natural power law antiderivatives on the right hand side series.]

b) What is its radius of convergence?

c) Use the Taylor series formula $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$ to evaluate the cubic Taylor approximation for this function and compare with the expression from part a) for its first 2 nonzero terms. [Hint: Use Maple to evaluate the necessary derivatives at $x=0$.]

d) Does this series converge at the endpoints of the interval of convergence? Explain.

Optional.

e) What does Maple give for the value of your part a) series formula if you first
> assume($a<x<b$)
where $a<x<b$ is the open interval of convergence?

▶ solution

$r=\frac{x^2}{4}$ geometric series — $\left|\frac{x^2}{4}\right|<1 \to |x|<2=R$ radius of convergence

① a) $$\frac{4}{4-x^2}=\frac{4}{4(1-\frac{x^2}{4})}=\frac{1}{1-\frac{x^2}{4}}=\sum_{n=0}^{\infty}\left(\frac{x^2}{4}\right)^n=\sum_{n=0}^{\infty}\frac{x^{2n}}{4^n}$$

$$\int\frac{4}{4-x^2}dx=\int\sum_{n=0}^{\infty}\frac{x^{2n}}{4^n}dx=\sum_{n=0}^{\infty}\int\frac{x^{2n}}{4^n}dx=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)4^n}+C_2$$

also for integrated series

$\ln\left(\frac{2+x}{2-x}\right)+C$ at $x=0$ $\underbrace{\ln\left(\frac{2}{2}\right)}_{=0}+C=0+C_2\to C=C_2$, subtract from both sides

$$f(x)=\boxed{\ln\left(\frac{2+x}{2-x}\right)=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)4^n}}$$

b) $\boxed{R=2}$ → interval: $-2\le x<2$

c) $f(0)=\ln\frac{2}{2}=\ln 1=0$, Maple: $f'(0)=1$, $f''(0)=0$, $f'''(0)=\frac{1}{2}$

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)x^3}{3!}$$
$$=0+x+0+\frac{1}{2\cdot(3\cdot 2)}x^3+\dots=\underbrace{x+\frac{1}{12}x^3}+\dots$$

$$\boxed{T_3(x)=x+\frac{1}{12}x^3}$$

d) $\left|\frac{x}{2}\right|=1\to\frac{x}{2}=\pm1$, $x=\pm2$ (endpoints)

$$\sum_{n=0}^{\infty}\frac{(\pm2)^{2n+1}}{(2n+1)4^n}=\sum_{n=0}^{\infty}\frac{(\pm1)^{2n}(\pm1)2^{2n}\cdot2}{(2n+1)4^n}=\sum_{n=0}^{\infty}\frac{\pm2}{2n+1}=\pm\infty\to\boxed{\text{diverges}}$$

overall sign, not alternating

$\frac{\pm2}{2n+1}\xrightarrow[n\to\infty]{}\pm\frac{1}{n}$ → same convergence properties as $p=1$ harmonic series (limit comparison test) which diverges

e) $$\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)4^n}=\boxed{2\,\mathrm{arctanh}\left(\frac{x}{2}\right)}$$
$$\left[=\ln\left(1+\frac{x}{2}\right)-\ln\left(1-\frac{x}{2}\right)=\ln\left(\frac{1+x/2}{1-x/2}\right)=\ln\left(\frac{2+x}{2-x}\right)\right]$$