MAT1505-03/04 17F Quiz 6 Print Name (Last, First) ________________, ______________|____

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation when appropriate). Indicate where technology is used and what type (Maple, GC).

1. a) The **logistic** probability distribution function is $P(x) = \dfrac{e^{\frac{\mu - x}{s}}}{s\left(e^{\frac{\mu-x}{s}}+1\right)^2}$, $s > 0$, for which $\int_{-\infty}^{\infty} P(x)\,dx = 1$. Introduce the standard variable $u = \dfrac{x-\mu}{s}$ to transform this integral to its standard form in terms of u (no need to evaluate it!).

b) For the particular such distribution function with $\mu = 2$, $s = 1$ show that this improper integral of part a) exists and has the stated value by expressing it as the sum of the integrals over the positive and negative half axes, and evaluating the necessary limits for these separate semi-infinite integrals, and then recombining them to obtain the final result. Note that a simple change of variable to the expression in parentheses in the integrand allows you to easily find the required antiderivative (but you can always check with Maple!).

c) For this particular distribution b), what is the probability that the random variable represented by this distribution function takes a value in the interval [0,4]? Give both the simplified exact value and its numerical value to 3 significant figures.

d) Suppose we set $\mu = 0$, $s = 1$, corresponding to the "standard variable" form for this family of distributions. Show that this function is even by setting $e^{-x} = \dfrac{1}{e^x}$ in $P(x)$ and simplifying, then comparing with $P(-x)$. Make a rough sketch of its plot in an appropriate window with one tickmark on the horizontal and vertical axes. [This function will be symmetric about its expected value for all values of the parameters since s only rescales the variable, not changing the shape of the distribution function, while μ just translates the peak left and right, not changing the shape.]

e) **Optional.** Evaluate the standard deviation σ (exactly and numerically) for the general distribution using: $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2\, P(x)\,dx$.

▶ solution

a) $$\int_{-\infty}^{\infty} \frac{e^{(\mu-x)/s}}{s\left(e^{(\mu-x)/s}+1\right)^2}\,dx = \boxed{\int_{-\infty}^{\infty}\frac{e^{-u}}{(1+e^{-u})^2}\,du}$$

$e^{(\mu-x)/s} \to e^{-u}$

$u = \frac{x-\mu}{s}$ $du = \frac{dx}{s}$

$x \to \pm\infty \to u \to \pm\infty$

b) $$\int \frac{e^{2-x}}{(e^{2-x}+1)^2}\,dx = \int \frac{-dw}{w^2} = -\int w^{-2}\,dw$$
$$= w^{-1} + C$$
$$= (e^{2-x}+1)^{-1}$$

$w = e^{2-x}+1$

$dw = -e^{2-x}dx$

$$\int_{-\infty}^{\infty}\frac{e^{2-x}dx}{(1+e^{2-x})^2} = \int_{-\infty}^{0}\frac{e^{2-x}dx}{(1+e^{2-x})^2} + \int_{0}^{\infty}\frac{e^{2-x}dx}{(1+e^{2-x})^2}$$

$$= \lim_{a\to-\infty}\int_a^0 \frac{e^{2-x}dx}{(1+e^{2-x})^2} + \lim_{a\to\infty}\int_0^a \frac{e^{2-x}dx}{(1+e^{2-x})^2}$$

$$= \lim_{a\to-\infty} \frac{1}{1+e^{2-x}}\Big|_a^0 + \lim_{a\to\infty}\frac{1}{1+e^{2-x}}\Big|_0^a$$

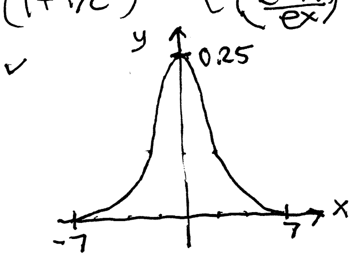
0 since $e^{2-a}\to\infty$

$$= \lim_{a\to-\infty}\left(\frac{1}{1+e^2} - \frac{1}{1+e^{2-a}}\right) = \frac{1}{1+e^2}$$
$$+ \lim_{a\to\infty}\left(\frac{1}{1+e^{2-a}} - \frac{1}{1+e^2}\right) \quad e^{2-a}\to 0 \qquad + \underbrace{1 - \frac{1}{1+e^2}}_{\frac{e^2}{1+e^2}}$$

$= 1$ ✓

c) $$P(0\le x\le 4) = \int_0^4 \frac{e^{2-x}}{(1+e^{2-x})^2}dx$$
$$= \frac{1}{1+e^{2-x}}\Big|_0^4 = \frac{1}{1+e^{-2}} - \frac{1}{1+e^2} = \frac{e^2}{e^2+1} - \frac{1}{1+e^2}$$
$$= \boxed{\frac{e^2-1}{e^2+1} \approx 0.762}$$

d) $$P(x) = \frac{e^{-x}}{(1+e^{-x})^2} = \frac{1/e^x}{(1+1/e^x)^2} = \frac{1}{e^x\left(\frac{e^x+1}{e^x}\right)^2}$$
$$= \frac{e^x}{(e^x+1)^2} = P(-x) \checkmark$$



e) Maple says

$\boxed{\sigma = \frac{\pi}{\sqrt{3}}s \approx 1.81 s}$