MAT1505-03/04 17F Quiz 4 Print Name (Last, First) ________________, ________________|____

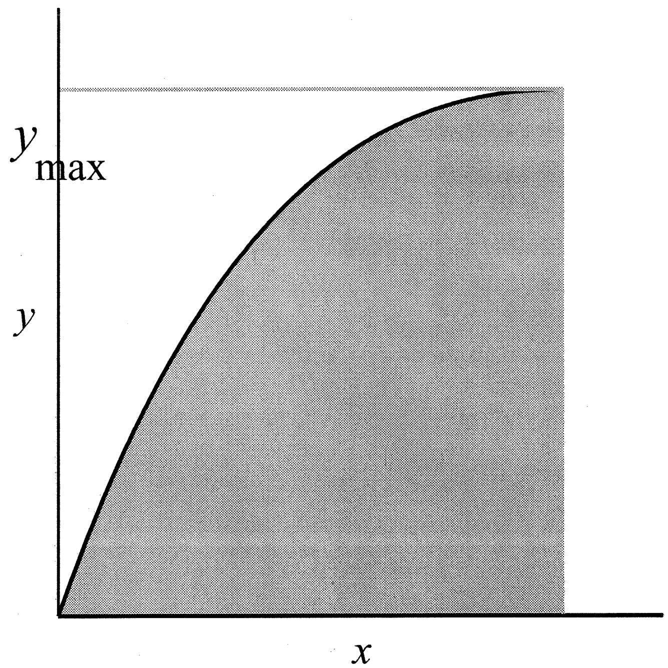
Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation when appropriate). Indicate where technology is used and what type (Maple, GC).

Consider the graph of $y = x\,e^{-\frac{x}{a}}$ for $a > 0$ on the interval $0 \le x \le x_{max}$ where x_{max} is the location of the maximum value of this function.

a) Determine exactly (x_{max}, y_{max}) and the area A_{rect} of the rectangle shown in the figure with this width and height.

b) Set up a definite integral representing the shaded area A and then evaluate it exactly by hand using integration by parts. Does your result agree with techology?

c) Evaluate and simplify the exact fractional area $\frac{A}{A_{rect}}$ and give its numerical value to 3 decimal places. Does your result seem consistent with the figure?



▶ solution

a) $y = x e^{-\frac{x}{a}}$

$\frac{dy}{dx} = 1 e^{-\frac{x}{a}} + x e^{-\frac{x}{a}}\left(-\frac{1}{a}\right)$

$= e^{-\frac{x}{a}}\left(1 - \frac{x}{a}\right) = 0 \to x = a$

$y = a e^{-\frac{a}{a}} = a e^{-1}$

so $\boxed{(x_{max}, y_{max}) = (a, a e^{-1})}$

$A_{rect} = x_{max}\, y_{max} = \boxed{a^2 e^{-1}}$

b) $A = \int_0^{x_{max}} y\,dx = \int_0^{x_{max}} x e^{-\frac{x}{a}}\,dx$

$= \boxed{\int_0^a \underbrace{x}_{u}\, \underbrace{e^{-\frac{x}{a}}\,dx}_{dv}}$

$du = dx \quad v = -a e^{-\frac{x}{a}}$

$= \underbrace{x}_{u}\underbrace{(-a e^{-\frac{x}{a}})}_{v}\Big|_0^a - \int \underbrace{-a e^{-\frac{x}{a}}}_{v}\, \underbrace{dx}_{du}$

$= \quad + a(-a) e^{-\frac{x}{a}} \Big|_0^a$

$= e^{-\frac{x}{a}}(-ax - a^2)\Big|_0^a = -2a^2 e^{-1} + a^2 e^0$

$= \boxed{a^2(1 - 2e^{-1})}$

Note: we could have first done a u-sub:

$\int_0^a x e^{-\frac{x}{a}}\,dx = a^2 \int_0^1 \frac{x}{a} e^{-\frac{x}{a}} \frac{dx}{a}$ ($x = a \to u = 1$, $x = 0 \to u = 0$)

$u = \frac{x}{a} \quad du = \frac{dx}{a}$

$= a^2 \underbrace{\int_0^1 u e^{-u}\,du}$

just some number, dependence on a factors out of integral!

c) $\frac{A}{A_{rect}} = \frac{a^2(1 - 2e^{-1})}{a^2 e^{-1}} = e(1 - 2e^{-1})$

$= \boxed{e - 2}$

$\approx \boxed{0.718}$

from memory! $e \approx 2.718$.