MAT1505-03/04 17f Quiz 2 Print Name (Last, First) ________________ , ________________|____

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC).

1. a) Find the total positive area between the graphs $y=\dfrac{x}{1+x^2}$, $y=\dfrac{1}{2}x$.

b) Repeat for the graphs $y=\dfrac{x}{1+x^2}$, $y=mx$, where $0\le m<1$.

c) Why must m satisfy the condition $0\le m<1$ for this problem to make sense? Explain.

d) Find the numerical value of m (to 6 decimal places) for which the total area is 1.

e) **Optional.** Find the exact value of m for which the total area is 1.

NOTE: ~~just~~ because a technology plot suggests $x=0,\pm 1$ is a soln of the intersection condition — you must justify it to show you can handle nonobvious scenarios!

▶ solution

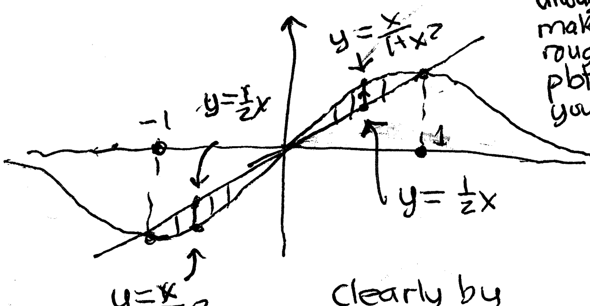
a) Find intersection points.

$$\frac{x}{1+x^2}=\frac{1}{2}x$$

$$2x=x(1+x^2)$$

$$0=x(1+x^2-2)=x(x^2-1)$$

$$x=0,\quad x=\pm 1$$



always make a rough plot if you can!

Clearly by symmetry:

$$A=\int_{-1}^{0}\frac{1}{2}x-\frac{x}{1+x^2}\,dx+\int_{0}^{1}\frac{x}{1+x^2}-\frac{1}{2}x\,dx$$

$$=2\int_0^1\left(\frac{x}{1+x^2}-\frac{1}{2}x\right)dx=2\left[\frac{1}{2}\ln(1+x^2)-\frac{1}{4}x^2\right]\Big|_0^1$$

$$\int\frac{x\,dx}{1+x^2}=\frac{1}{2}\int\frac{du}{u}=\frac{1}{2}\ln|u|+C=\frac{1}{2}\ln(1+x^2)+C$$

$x\,dx=\frac{1}{2}du$

$u=1+x^2$

$du=2x\,dx$

$$=2\left(\frac{1}{2}\ln 2-\frac{1}{4}-\frac{1}{2}\cancel{\ln 1}^{\,0}\right)$$

$$=\boxed{\ln 2-\frac{1}{2}}$$

$$\approx \boxed{0.193147}$$

b)

$$\frac{x}{1+x^2}=mx$$

$$x=mx(1+x^2)=x(m+mx^2)$$

$$0=x(m-1+mx^2)$$

$$x=0,\quad x^2=\frac{1-m}{m}\to x=\pm\sqrt{\frac{1-m}{m}}$$

$$A=2\int_0^{\sqrt{(1-m)/m}}\left(\frac{x}{1+x^2}-mx\right)dx$$

$$=2\left(\frac{1}{2}\ln(1+x^2)-\frac{m}{2}x^2\right)\Big|_0^{\sqrt{(1-m)/m}}$$

$$=\ln\Big(\underbrace{1+\frac{1-m}{m}}_{\frac{1}{m}}\Big)-\frac{m}{2}\left(\frac{1-m}{\cancel{m}}\right)-\cancel{\ln 1}^{\,0}$$

$$=\boxed{\ln\left(\frac{1}{m}\right)-(1-m)}$$

difference of both positive terms in this form

c) real solutions $x=\pm\sqrt{\frac{1-m}{m}}$

only exist if $m\le 1$ or $m>0$

but $m=1$ gives again the zero solution.

Thus only for proper fraction positive m is the finite region between the two curves.

($m=0$ leads to an infinite region with infinite area!)

d) solve numerically for $0<m<1$: $\ln\frac{1}{m}-(1-m)=1$

Find (Maple): $\boxed{m\approx 0.158594}$

e) Maple gives exactly: $\boxed{-\text{LambertW}(-e^{-2})}$, $-\text{LambertW}(-1,-e^{-2})$

≈ 0.1585943396, $3.14\cancel{61}93221$

since $0\le m<1$.