MAT1505-03/04 17F Test 3 Answers (1)

(1) a) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{4}{2(n+1)+1\;2^{2(n+1)+1}}\cdot\frac{(2n+1)\,2^{2n+1}}{4}$

$= \lim_{n\to\infty}\left(\frac{2n+1}{2n+3}\right)2^{\underbrace{2n+1-(2n+3)}_{=-2}} = \lim_{n\to\infty}\frac{1}{4}\left(\frac{2n+1}{2n+3}\right)$

$= \frac{1}{4} < 1$ converges by absolute convergence ratio test

b) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{4}{2n+3}\frac{|2x-1|^{2n+3}}{2^{2n+3}}\cdot\frac{2n+1}{4}\frac{2^{2n+1}}{|2x-1|^{2n+1}}$

$= \lim_{n\to\infty}\frac{|2x-1|^2}{2^2}\left(\frac{2n+1}{2n+3}\right) = \lim_{n\to\infty}\frac{|2x-1|^2}{4}\left(\frac{2n}{2n}\right)$

$= \frac{|2x-1|^2}{4} < 1$ for convergence

$\frac{|2x-1|}{2} < 1 \to \underset{\uparrow\ \text{center}}{\left|x-\tfrac{1}{2}\right|} < \underset{\uparrow\ \text{radius}}{\boxed{1 = R}}$

$x-\frac{1}{2} = 1 \to x = 3/2$ $\qquad -\frac{1}{2} < x < \frac{3}{2}$

$\phantom{x-\frac{1}{2}} = -1 \to x = -1/2$ $\qquad \frac{1}{2}-1=-\frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{2}+1 = 3/2$

$2x-1 = 2:\ \sum_{n=0}^{\infty}(-1)^n\frac{4(2)^{2n+1}}{(2n+1)2^{2n+1}} = \sum_{n=0}^{\infty}(-1)^n\frac{4}{2n+1}$ converges (decreasing abs val, alternating sign)

$ = -2:\ \sum_{n=0}^{\infty}(-1)^n\frac{4(-2)^{2n+1}}{(2n+1)2^{2n+1}} = \sum_{n=0}^{\infty}(-1)^n\frac{(-4)}{2n+1}$ converges also (just overall sign change)

interval of convergence: $\boxed{-\frac{1}{2}\le x\le\frac{3}{2} \text{ or } [-1/2, 3/2]}$.

c) $\sum_{n=0}^{\infty} a_n$:

$a_6 = .0000375 > 10^{-5}$

$a_7 = -8.13\cdot10^{-6} < 10^{-5}$, $\sum_{n=0}^{6} a_n = 1.854597 \approx 1.85460$ = approx

$\sum_{n=0}^{\infty} a_n = 1.854590 \approx 1.85459$ to 5 decimal places

= exact $\rightarrow$ $|\text{exact}-\text{approx}| = 6.67\times10^{-6} < 10^{-5}$ ✓

$\boxed{\text{Yes, within estimate. First 7 terms needed}}$

(2) a) $\int_2^{\infty}\frac{dn}{n(\ln n)^p} = \int_{\ln 2}^{\infty}u^{-p}\,du = \lim_{a\to\infty}\int_{\ln 2}^{a}u^{-p}\,du$

$u = \ln n$, $du = \frac{dn}{n}$

$n=2 \to u = \ln 2$

$n\to\infty \quad u\to\infty$

$p\neq1$: $= \lim_{a\to\infty}\frac{u^{1-p}}{1-p}\Big|_{\ln 2}^{a} = \lim_{a\to\infty}\underbrace{\frac{a^{1-p}}{1-p}}_{\to 0} - \frac{(\ln 2)^{1-p}}{1-p}$

[$p>1$ so $1-p<0$ to put a in denominator]

$= -\frac{(\ln 2)^{1-p}}{1-p}$ finite, so the corresponding series converges if $\boxed{p>1}$

but if $p=1$: $\int_2^{\infty}\frac{dn}{n\ln n} = \int_{\ln 2}^{\infty}u^{-1}\,du$

$= \lim_{a\to\infty}\int_{\ln 2}^{a}u^{-1}\,du = \lim_{a\to\infty}\ln u\Big|_{\ln 2}^{a}$

$= \lim_{a\to\infty}\underbrace{\ln a}_{\to\infty} - \ln\ln 2 = \infty$ diverges if $p=1$

so original series diverges

(3) a) $(1-k^2\sin^2x)^{-1/2} = 1-\frac{1}{2}(-k^2\sin^2x) - \frac{\frac{1}{2}\left(-\frac{3}{2}\right)}{1\cdot2}(-k^2\sin^2x)^2 + \dots$

$= 1+\frac{k^2}{2}\sin^2x + \frac{3}{8}k^4\sin^4x + \dots$

$\int_0^{\pi/2}(1-k^2\sin^2x)^{-1/2}\,dx = \underbrace{\int_0^{\pi/2}1\,dx}_{\frac{\pi}{2}} + \frac{k^2}{2}\underbrace{\int_0^{\pi/2}\sin^2x\,dx}_{\frac{\pi}{4}} + \frac{3k^4}{8}\underbrace{\int_0^{\pi/2}\sin^4x\,dx}_{\frac{3\pi}{16}} + \dots$

(3) a) continued

$= \frac{\pi}{2}\left(1+\frac{k^2}{4}+\frac{9}{64}k^4\right)+\dots$

$T = 4\sqrt{\frac{L}{g}}\,\frac{\pi}{2}(\dots) = \underbrace{2\pi\sqrt{\frac{L}{g}}}_{T_0}(\dots)$

$\boxed{T = T_0\left(1+\frac{1}{4}k^2+\frac{9}{64}k^4+\dots\right)}$, $\boxed{T_0 = 2\pi\sqrt{\frac{L}{g}}}$

b) $\frac{T}{T_0} = 1 + 0.016768 + 0.000631 + \dots$

$= \underbrace{\boxed{1.017378}}_{\equiv\,\text{approx}_3} + \dots$ need decimal values for interpretation!

$\boxed{\frac{T}{T_0} = \frac{2\,\text{EllipticK}(k)}{\pi} \approx 1.017409}$ = exact.

$\frac{\text{approx}_3-\text{exact}}{\text{exact}} \approx \boxed{-0.00003041}$ $\boxed{\approx -0.003\%}$ 3 terms

$\frac{\text{approx}_2-\text{exact}}{\text{exact}} \approx -0.000651$ $\boxed{\approx -0.065\%}$ 2 terms

$\frac{\text{approx}_1-\text{exact}}{\text{exact}} \approx -0.0171$ $\boxed{\approx -1.7\%}$ 1 term

note series converges to $4\arctan\left(x-\frac{1}{2}\right)$

$x-\frac{1}{2} = \pm1 \longrightarrow 4\arctan(\pm1) = \pm\pi$

then set $2x-1 = 1$ $(x=1)$

get $4\arctan\frac{1}{2}$

MAT1505-03/04 17F Test 3 Answers (2)

4) a) $$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(2n+2)!}{((n+1)!)^2}\frac{|x|^{2n+3}}{(2n+3)4^{n+1}}\frac{(n!)^2(2n+1)4^n}{(2n)!\,|x|^{2n+1}} = \frac{(2n+2)(2n+1)\cancel{(2n)!}}{((n+1)\cancel{n!})^2}\frac{(2n+1)}{(2n+3)}\frac{|x|^2}{4}\frac{\cancel{(n!)^2}}{\cancel{(2n)!}}$$

$$= \frac{(2n+2)(2n+1)^2}{(n+1)^2\,4(2n+3)}|x|^2 = \overbrace{\frac{(2n+1)^2}{2(2n+3)(n+1)}}^{\to 1 \text{ as } n\to\infty}|x|^2 \xrightarrow{n\to\infty} |x|^2 < 1 \to |x| < \boxed{1 = R}$$

b) $x=1$: $$\underbrace{\arcsin(1)}_{=\frac{\pi}{2}} = \sum_{n=0}^{\infty} \underbrace{\frac{(2n)!}{(n!)^2 4^n(2n+1)}}\ \overset{n\to\infty}{\rightsquigarrow}\ \frac{\sqrt{2\pi(2n)}\left(\frac{2n}{e}\right)^{2n}}{\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right)^2 4^n(2n+1)}$$

$$= \frac{\sqrt{2}}{\sqrt{2\pi n}}\frac{\cancel{2^{2n}}\cancel{\left(\frac{n}{e}\right)^{2n}}}{\cancel{4^n}\cancel{\left(\frac{n}{e}\right)^{2n}}\,2n+1} = \frac{1}{\sqrt{\pi}\sqrt{n}(2n+1)} \rightsquigarrow \frac{1}{2\sqrt{\pi}\,n^{3/2}}$$

$p = 3/2 > 1$ convergent p-series.

c) $$\arcsin x = \int_0^x (1+(-t^2))^{-1/2}\,dt = \int_0^x 1 - \tfrac12(-t^2) - \frac{\frac12\left(-\frac32\right)}{1\cdot 2}(-t^2)^2 + \cdots\,dt$$

$$= \int_0^x 1 + \tfrac12 t^2 + \tfrac38 t^4 + \cdots\,dt = t + \tfrac16 t^3 + \tfrac{3}{40}t^5 + \cdots\Big|_0^x = \boxed{x + \tfrac16 x^3 + \tfrac{3}{40}x^5} + \cdots$$

$$\sum_{n=0}^{\infty}\frac{(2n)!\,x^{2n+1}}{(n!)^2 4^n(2n+1)} = \frac{0!\,x}{(0!)^2 4^0\, 1} + \frac{2!\,x^3}{(1!)^2 4^1\, 3} + \frac{4!\,x^5}{(2!)4^2\, 5} + \cdots = x + \tfrac16 x^3 + \tfrac{3}{40}x^5 + \cdots$$

$$= \frac{4\cdot 3\cdot 2}{2\cdot 4^2\cdot 5} = \frac{3}{40}$$

agreement!

d) optional $(1-t^2)^{-1/2}$

$$= 1 - \tfrac12(-t^2) - \frac{\frac12\left(-\frac32\right)}{2!}(-t^2)^2 + -\frac{\frac12\left(-\frac32\right)\left(-\frac52\right)}{3!}(-t^2)^3 - \frac{\frac12\left(-\frac32\right)\left(-\frac52\right)\left(-\frac72\right)}{4!}(-t^2)^4 + \cdots$$

$$1 + \tfrac12 t^2 + \frac{1\cdot2\cdot3\cdot4}{2^2\cdot 2(1)2(2)}\frac{t^4}{2!} + \frac{1\cdot2\cdot3\cdot4\cdot5\cdot6}{2^3\cdot 2(1)2(2)2(3)}\frac{t^6}{3!} + \frac{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8}{2^4\, 2(1)2(2)2(3)2(4)}\frac{t^8}{4!} + \cdots = \sum_{n=0}^{\infty}\frac{(2n)!}{(n!)^2 4^n}t^{2n}$$

$(2\cdot2)!$, 2^2, $2!$, 4^2; 2^3, $2!$, 4^3, $(2\cdot3)!$; $(2\cdot4)!$, 2^4, $4!$, 4^4

$$\int_0^x (1-t^2)^{-1/2}dt = \int_0^x \sum_{n=0}^{\infty}\frac{(2n)!}{(n!)^2 4^n}t^{2n}\,dt = \sum_{n=0}^{\infty}\frac{(2n)!}{(n!)^2 4^n}\underbrace{\int_0^x t^{2n}dt}_{\frac{t^{2n+1}}{2n+1}\Big|_0^x = \frac{x^{2n+1}}{2n+1}} = \sum_{n=0}^{\infty}\frac{(2n)!\,x^{2n+1}}{(n!)^2(2n+1)}\ \checkmark$$

e) $\arcsin(0.25) = \sum_{n=0}^{\infty} b_n$, $b_5 = 5.3\times10^{-9}$, $b_6 = 2.6\,10^{-10}$, $\underbrace{b_7 = 1.3\,10^{-11}}_{\text{seems unnecessary for 10 dec place accuracy.}}$

$\sum_{n=0}^{6} b_n \approx 0.2526802552$

$\arcsin(0.25) = 0.2526802551$ (maple)

differ in last digit at Digits = 10 $\to$ increase to Digits = 12 and they agree with Maple's numerical value to 10 digits.

↑ working at 10 digits, can't trust 10th digit in sum due to truncation error

5) $$E = \frac{\delta}{R^3} - \frac{\delta}{(R+d)^3} = \frac{\delta}{R^3} - \frac{\delta}{(R(1+d/R))^3} = \frac{\delta}{R^3} - \frac{\delta}{R^3}\left(1+\frac{d}{R}\right)^{-3} = \frac{\delta}{R^3}\left(1-\left(1+\frac{d}{R}\right)^{-3}\right)$$ binomial expansion

$$= \frac{\delta}{R^3}\left(1 - \left(1 - 3\left(\frac{d}{R}\right) - \frac{3(-4)}{1\cdot 2}\left(\frac{d}{R}\right)^2 + \cdots\right)\right) = \frac{\delta}{R^3}\left(3\frac{d}{R} - 6\frac{d^2}{R^2} + \cdots\right) = \boxed{\frac{3d\delta}{R^4}} + \cdots$$

inverse quartic

or linear approx: $E(d) = \frac{\delta}{R^3} - \frac{\delta}{(R+d)^3} \approx E(0) + E'(0)d = 0 + \left(\frac{3\delta}{R^4}\right)d = \frac{3d\delta}{R^4}$