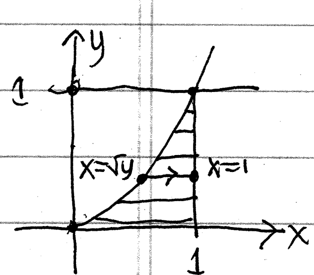
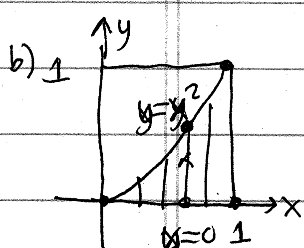
MAT2500-01/04 16S Test 3 Answers (1)

① a) $\displaystyle\int_{y=0}^{1}\int_{x=\sqrt{y}}^{1} \frac{y e^{x^2}}{x^3}\,dx\,dy$



$x=\sqrt{y}..1$ while $y=0..1$



$y=0..x^2$ while $x=0..1$

$$\int_0^1\int_0^{x^2} \frac{y e^{x^2}}{x^3}\,dy\,dx$$

$$= \frac{y^2 e^{x^2}}{2x^3}\Big|_{y=0}^{y=x^2} = \frac{x^4}{2x^3}e^{x^2} = \frac{x}{2}e^{x^2}$$

$$= \int_0^1 \frac{x}{2}e^{x^2}\,dx = \int_{x=0}^{x=1} \frac{1}{4}e^u\,du$$

$u=x^2$
$du=2x\,dx$

$$= \frac{1}{4}e^u\Big|_{x=0}^{x=1} = \frac{1}{4}e^{x^2}\Big|_0^1 = \frac{1}{4}(e^1-e^0)$$

$$= \boxed{\frac{1}{4}(e-1)} \approx 0.4296$$

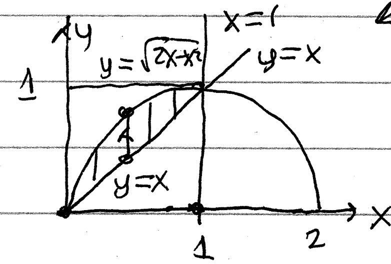
Maple gets this exact result for both iterations.

② $\displaystyle\int_{x=0}^{1}\int_{y=x}^{\sqrt{2x-x^2}} xy\,dy\,dx$

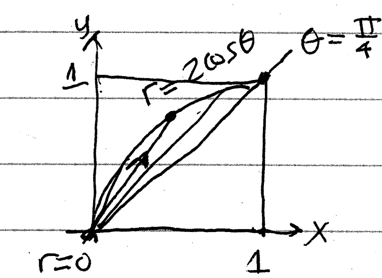
$y^2 = 2x - x^2$
$x^2+y^2 = 2x \to r^2 = 2r\cos\theta$
$r = 2\cos\theta$

$x^2-2x+y^2=0$
$(x-1)^2-1+y^2=0$
$(x-1)^2+y^2=1$ circle radius 1, center (1,0)

a)



$y=x..\sqrt{2x-x^2}$ while $x=0..1$



$r=0..2\cos\theta$
while $\theta=\frac{\pi}{4}..\frac{\pi}{2}$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{2\cos\theta} (r\cos\theta)(r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{2\cos\theta} r^3 \sin\theta\cos\theta\,dr\,d\theta$$

$$\frac{r^4}{4}\sin\theta\cos\theta\Big|_{r=0}^{r=2\cos\theta} = \frac{16\cos^4\theta\sin\theta\cos\theta}{4}$$

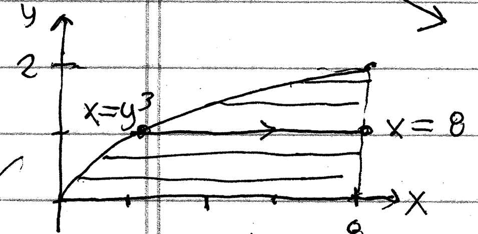
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 4\underbrace{\cos^5\theta}_{u^5}\,\underbrace{\sin\theta\,d\theta}_{-du} = -4\frac{u^6}{6}\Big|_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}}$$

$$= -\frac{2}{3}\cos^6\theta\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{2}{3}\left(\frac{1}{2^{1/2}}\right)^6 = \frac{2}{3\cdot 2^3} = \boxed{\frac{1}{12}}$$

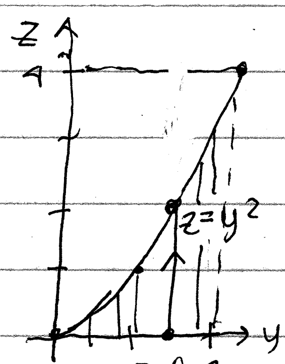
Maple gets this exact result for both iterations.

MAT2500-01/04 16 S Test 3 Answers (2)

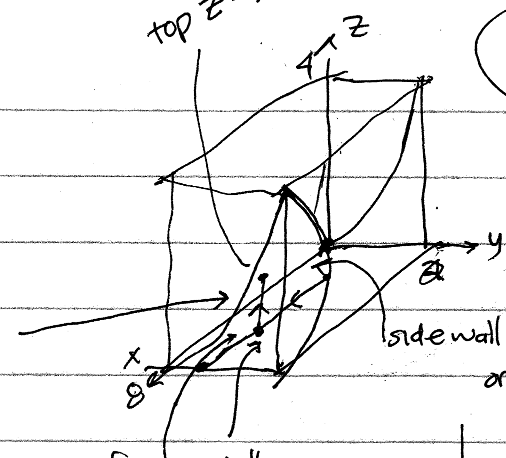
③ a) $\underbrace{\int_{y=0}^{y=2}\int_{x=y^3}^{x=8}}\int_{z=0}^{z=y^2} x\,y\,z \; dz\,dx\,dy$



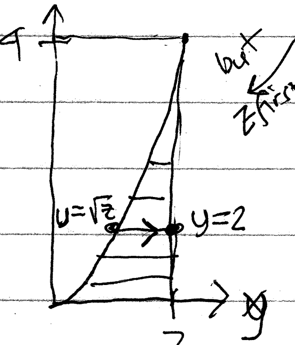
$x=8..y^3$ while $y=0..2$



$\underbrace{y=0..2}$

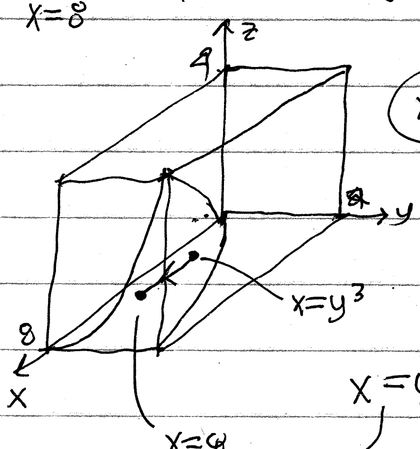


z first



but x ← z first

↑ projects onto yz plane



x first

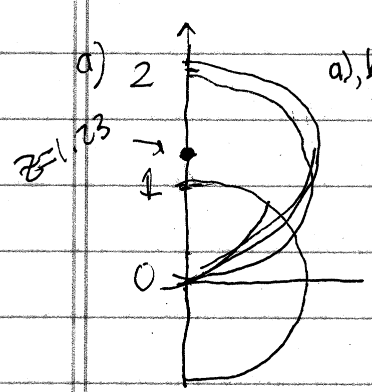
$x=y^3..8$

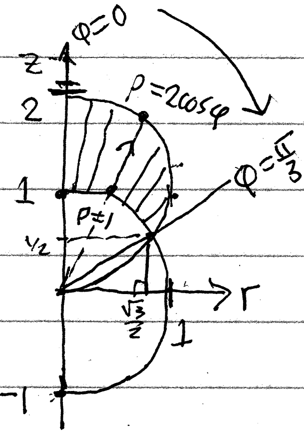
so $y=\sqrt{z}..2$ while $z=0..4$

$$\boxed{\int_0^4\int_{\sqrt{z}}^{2}\int_{y^3}^{8} x\,y\,z\;dx\,dy\,dz}$$

b) Maple evaluates both to $\boxed{\frac{256}{3}}$ so they agree.

④





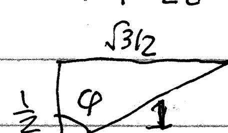
a) $r^2+z^2=1$ $[\rho=1]$ center at origin

$r^2+(z-1)^2=1$ center at $z=1$

$r^2+z^2-2z+1=1$ $\qquad \rho=2\cos\varphi$

$r^2+z^2=2z \rightarrow [\rho^2=2\rho\cos\varphi]$

$1=2z \rightarrow z=\frac{1}{2} \rightarrow r=\sqrt{1-z^2}=\sqrt{1-\frac{1}{4}}=\frac{\sqrt{3}}{2}$



$\varphi=\arccos\frac{1}{2}=\frac{\pi}{3}$

so $\rho=1..2\cos\varphi$ while $\varphi=0..\pi/3$ (and $\theta=0..2\pi$!)

c) $V=\int_0^{2\pi}\int_0^{\pi/3}\int_0^{2\cos\varphi}\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$

$M_{xy}=\iiint_R \overset{\rho\cos\varphi}{z}\,dV=\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^{2\cos\varphi}\underbrace{\rho\cos\varphi\,\rho^2\sin\varphi}_{\rho^3\cos\varphi\sin\varphi}\,d\rho\,d\varphi\,d\theta$

MAT2500-01/04 16S Test 3 Answers

[NOTE! $\frac{\sqrt{3}}{2} = \sin\pi/3$, $1/2 = \cos\pi/3$]

④ d) $V = \underbrace{\int_0^{2\pi} 1\, d\theta}_{2\pi} \int_0^{\pi/3} \underbrace{\int_1^{2\cos\varphi} \rho^2 \sin\varphi\, d\rho}\, d\varphi = \frac{2\pi}{3} \int_0^{\frac{\pi}{3}} (\underbrace{8\cos^3\varphi - 1}_{8u^3-1}) \underbrace{\sin\varphi\, d\varphi}_{-du}$

$\frac{\rho^3}{3}\sin\varphi \Big|_1^{2\cos\varphi} = \frac{(8\cos^3\varphi - 1)\sin\varphi}{3}$

$\int 1 - 8u^3 du = u - \frac{8}{4}u^4$

$= \frac{2\pi}{3} \cos\varphi\,(1 - 2\cos^3\varphi) \Big|_0^{\pi/3} = \frac{2\pi}{3}\Big[\underbrace{\underbrace{(\tfrac{1}{2})(1 - 2\tfrac{1}{8})}_{3/4} - \underbrace{1(1-2)}_{+1}}_{\frac{11}{8}}\Big] = \boxed{\frac{11\pi}{12}}$

$M_{xy} = \underbrace{\int_0^{2\pi} 1\, d\theta}_{2\pi} \int_0^{\frac{\pi}{3}} \underbrace{\int_0^{2\cos\varphi} \underbrace{(\rho\cos\varphi)\,\rho^2 \sin\varphi}_{\rho^3\cos\varphi\sin\varphi}\, d\rho}\, d\varphi = \frac{\pi}{2} \int_0^{\frac{\pi}{3}} (\underbrace{16\cos^4\varphi - 1}_{16u^4-1}) \underbrace{\cos\varphi}_{u} \underbrace{\sin\varphi\, d\varphi}_{(-du)}$

$\frac{\rho^4}{4}\cos\varphi\sin\varphi \Big|_{\rho=1}^{\rho=2\cos\varphi} = \frac{16\cos^4\varphi - 1}{4}\cos\varphi\sin\varphi$

$\int u - 16u^5 du$
$\frac{u^2}{2} - \frac{16u^6}{6}$

$= \left(\frac{\pi}{2}\right) \frac{\cos^2\varphi}{2}\left(1 - \frac{16}{3}\cos^4\varphi\right)\Big|_0^{\pi/3} = \frac{\pi}{4}\Big(\underbrace{(\tfrac{1}{4})(1 - \tfrac{16}{3}\tfrac{1}{16})}_{} - \underbrace{(1 - \tfrac{16}{3})}_{+\frac{13}{3}}\Big) = \frac{\pi}{4}\left(\frac{1}{6} + \frac{13}{3}\right) = \frac{\pi}{4}\frac{27}{6}$

$= \boxed{\frac{9\pi}{8}}$

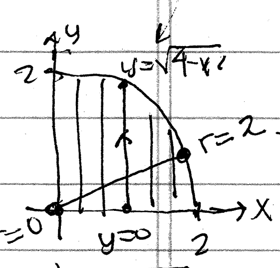
$\bar{z} = \frac{M_{xy}}{V} = \frac{9\pi/8}{11\pi/12} = \frac{\overset{3}{\cancel{12}}}{11}\frac{9}{\cancel{8}_2} = \boxed{\frac{27}{22}} \approx 1.227 \approx \boxed{1.23}$ 2 decimal places.

about 1/4 way up from lower north pole to upper north pole seems reasonable

⑤ a) $\int_{x=0}^{x=2} \int_{y=0}^{y=\sqrt{4-x^2}} \int_{z=0}^{z=4-x^2-y^2} \underbrace{\sqrt{x^2+y^2}}_{=r}\, dz\, dy\, dx = \boxed{\int_0^{\frac{\pi}{2}} \int_0^2 \int_0^{4-r^2} \underbrace{(r)\,r}_{=r^2}\, dz\, dr\, d\theta}$

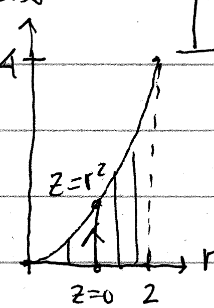
$z = 4 - r^2$

$y^2 = 4 - x^2$
$x^2 + y^2 = 4$
$r = 2$



$x = 0..2$

$r=2 \to \theta = 0..\pi/2$



b) $r^2 z \Big|_{z=0}^{z=4-r^2}$

$\int_0^2 4r^2 - r^4\, dr$

$= \frac{4}{3}r^3 - \frac{r^5}{5}\Big|_0^2$

$= 32\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{64}{15}$

$\int_0^{\frac{\pi}{2}} \frac{64}{15} d\theta = \frac{\pi}{2}\left(\frac{64}{15}\right) = \boxed{\frac{32\pi}{15}}$

$\boxed{\approx 6.702}$

Maple gives this numerical value for original integral which it cannot integrate exactly.